# GRAVITATION – ALL DERIVATIONS

## Acceleration due to gravity

Consider a small body of mass m kept near the surface of earth. The weight of this body is given by

W = mg

Also, the gravitational force of earth on the body is

$F = \frac{GMm}{R^2}$, where M is the mass of the earth and R is the radius of the earth. (As the size of body is very small as compared to size of earth, we have considered the distance between the centre of body and earth as radius of earth).

Therefore, we can write

$$mg = \frac{GMm}{R^2}$$

$\Rightarrow \boxed{g = \frac{GM}{R^2}}$ which is the formula for acceleration due to gravitation.

As $M = 6\times10^{24}\,kg, R = 6.4\times10^{6}\,m$, putting all these values, we get g = 9.8 $ms^{-2}$.

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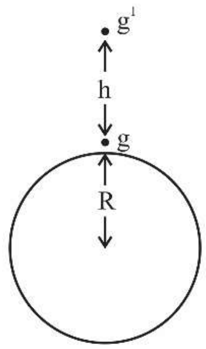
## Variation of acceleration due to gravity with height

As acceleration due to gravity at surface of earth, $g = \frac{GM}{R^2}$. Therefore, acceleration due to gravity at a height h above the surface of earth $g' = \frac{GM}{(R+h)^2}$.




$$\therefore \frac{g'}{g} = \frac{\frac{GM}{(R+h)^2}}{\frac{GM}{R^2}}$$

$$\Rightarrow \frac{g'}{g} = \frac{R^2}{(R+h)^2} \qquad \text{. . . . . . . . . . (i)}$$

$$\Rightarrow \frac{g'}{g} = \frac{R^2}{R^2(R+\frac{h}{R})^2}$$

$$\Rightarrow \frac{g'}{g} = \frac{1}{(R+\frac{h}{R})^2}$$

$$\Rightarrow \frac{g'}{g} = \left(1+\frac{h}{R}\right)^{-2}$$

If h << R, We can expand above expression using binomial theorem

$$\frac{g'}{g} = \left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow 1 - \frac{g'}{g} = \frac{2h}{R}$$

$$\Rightarrow \frac{g - g'}{g} \times 100 = \frac{2h}{R} \times 100$$

i.e. percentage decrease in the value of g at a height h = $\frac{2h}{R} \times 100$. Where h is very small as compared to radius of earth.

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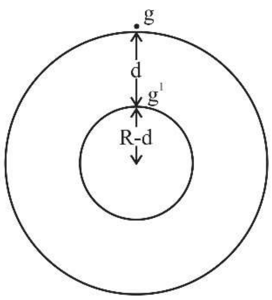
## Variation of g with depth

If body is taken at a depth d below the surface of earth then, $g' = \frac{GM'}{(R-d)^2}$.

Where $M'$ is the mass of that spherical part of earth whose radius is $(R-d)$.




Let earth be a uniform sphere of density ρ, then

$$M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore g = \frac{G}{R^2}\frac{4}{3}\pi R^3 \rho$$

$$\Rightarrow g = \frac{4}{3} G\pi R\rho$$

And,

$$M' = \frac{4}{3}\pi (R-d)^3 \rho$$

$$\therefore g' = \frac{G}{(R-d)^2}\frac{4}{3}\pi (R-d)^3 \rho$$

$$\Rightarrow g' = \frac{4}{3} G\pi (R-d)\rho$$

Therefore,

$$\frac{g'}{g}=\frac{\frac{4}{3}G\pi(R-d)\rho}{\frac{4}{3}G\pi R\rho}$$

$$\Rightarrow \frac{g'}{g}=\frac{R-d}{R} \Rightarrow \frac{g'}{g}=1-\frac{d}{R}$$

Hence, percentage decrease at a depth d in the value of g is given by $\left(1-\frac{d}{R}\right)\times 100$.

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## Variation in the value of g due to shape of earth

Since earth is not perfectly spherical in shape and radius at poles is less than the radius of earth at equator, therefore there is a slight difference in the value of g on these two places.

If $R_P$ = Radius of earth at poles and $g_P$ = acceleration due to gravity at poles and

If $R_E$ = Radius of earth at equator and $g_E$ = acceleration due to gravity at equator

Then, $g_P=\frac{GM}{R_P^2}$ and $g_E=\frac{GM}{R_E^2}$. Since $R_P<R_E \therefore g_P>g_E$.

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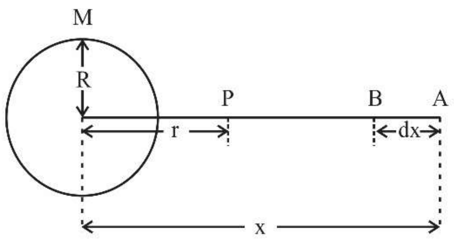
## Gravitational potential at a distance r due to a body of mass M

Consider a body of mass M. A body of mass m is place at a distance $x$ from the centre of this body.




Then force of gravitation between them is, $F=\frac{GMm}{x^2}$ .

Small amount of work (dW) done to move mass m through small distance d$x$ towards mass M is

$$dW = Fdx$$

$$\Rightarrow dW=\frac{GMm}{x^2}dx$$

Therefore, total work done to move this body from x = ∞ to $x$ = r is given by

$$\int dW = \int_{\infty}^{r} \frac{GMm}{x^2} dx$$

$$\Rightarrow W = GMm \int_{\infty}^{r} \frac{1}{x^2} dx$$

$$\Rightarrow W = GMm \left[ \frac{x^{-2+1}}{-2+1} \right]_{\infty}^{r}$$

$$\Rightarrow W = -GMm \left[ \frac{1}{x} \right]_{\infty}^{r}$$

$$\Rightarrow W = -GMm \left[ \frac{1}{r} - \frac{1}{\infty} \right]$$

$$\Rightarrow W = -\frac{GMm}{r}$$

By definition, potential at P is $V = \frac{W}{m}$. Therefore,

$$\boxed{V = -\frac{GM}{r}}$$

Clearly, the work done to arrange this system is $W = -\frac{GMm}{r}$. This work is stored in the system in the form of potential energy.

In general, potential energy of a system of two bodies of masses $m_1$ and $m_2$ placed at a distance r apart is given by

$$\boxed{U = -\frac{Gm_1m_2}{r}}.$$

## Prove that the work done to move a body of mass m through a distance h against the gravitational force is mgh.

For a body of mass m placed at the surface of earth, the potential energy of the system is given by

$$U_1 = -\frac{GMm}{R}$$

Now, if this body is taken to a height h, potential energy is given by

$$U_2 = -\frac{GMm}{R+h}$$

Change in potential energy is $\Delta U = U_2 - U_1$

$$\Rightarrow \Delta U = -\frac{GMm}{R+h} + \frac{GMm}{R}$$

$$\Rightarrow \Delta U = \frac{GMm}{R} \left( 1 - \frac{1}{1 + \frac{h}{R}} \right)$$

$$\Rightarrow \Delta U = \frac{GMm}{R}\left(1-\left(1+\frac{h}{R}\right)^{-1}\right)$$

For small heights, above expression can expanded binomially and it can be written as

$$\Rightarrow \Delta U = \frac{GMm}{R}\left(1-\left(1-\frac{h}{R}\right)\right)$$

$$\Rightarrow \Delta U = \frac{GMm}{R}\left(\frac{h}{R}\right)$$

$$\Rightarrow \Delta U = \frac{GMmh}{R^2} \Rightarrow \boxed{\Delta U = mgh}$$

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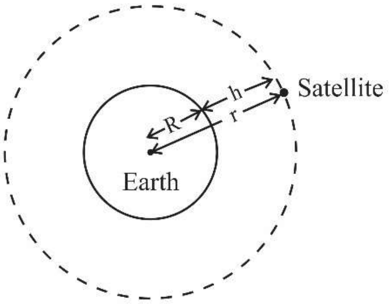
## Satellite

A body that revolves around a planet is called a satellite. Satellite is of two types: artificial and natural.

We launch a satellite around earth in an orbit with such a speed that centripetal force required by satellite is equal to the gravitational force of earth in that orbit.

Consider a satellite of mass m revolving around earth at a height h from the surface of earth. If orbital radius of satellite is r then clearly r = R + h.




The various parameters of the satellite can be obtained as:

### Orbital velocity

As, centripetal force = Gravitational force

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{GM}{r}}}$$

$$\Rightarrow v = \sqrt{\frac{GM}{R+h}}$$

If satellite is revolving very close to earth, then we can write above expression as

$$v = \sqrt{\frac{GM}{R}}$$

Putting all the values in the above expression, we get

$v = 7.92\ kms^{-1}$.

## Time period

It is the time taken by satellite to complete one orbit around earth.

$$T = \frac{2\pi r}{v}$$

$$\Rightarrow T = \frac{2\pi r}{\sqrt{\dfrac{GM}{r}}}$$

$$\Rightarrow \boxed{T = 2\pi r\sqrt{\frac{r}{GM}}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

If $h << R$, then

$$T = 2\pi\sqrt{\frac{R^3}{GM}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{R^3}{G\frac{4}{3}\pi R^3\rho}}$$

$$\Rightarrow T = \sqrt{\frac{4\pi^2}{G\frac{4}{3}\pi R^3\rho}}$$

$$\Rightarrow \boxed{T = \sqrt{\frac{3\pi}{G\pi}}}$$

## Height of satellite

As $T = 2\pi\sqrt{\dfrac{(R+h)^3}{GM}}$

Squaring both sides, we get

$$T^2 = 4\pi^2\frac{(R+h)^3}{GM}$$

$$\Rightarrow \frac{T^2 GM}{4\pi^2} = (R+h)^3$$

$$\Rightarrow R + h = \left(\frac{T^2 GM}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \boxed{h = \left(\frac{T^2 GM}{4\pi^2}\right)^{\frac{1}{3}} - R}$$

### Kinetic Energy

$$KE = \frac{1}{2}mv^2 \Rightarrow KE = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2$$

$$\Rightarrow KE = \frac{1}{2}\frac{GMm}{r}$$

### Potential energy

$$PE = -\frac{GMm}{r}$$

### Total energy

$$TE = KE + PE$$

$$TE = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$\Rightarrow TE = -\frac{1}{2}\frac{GMm}{r}$$

### Binding Energy

It is defined as the amount of energy required to remove a satellite from its orbit. It is equal to the positive of total energy of satellite.

$$BE = \frac{1}{2}\frac{GMm}{r}.$$

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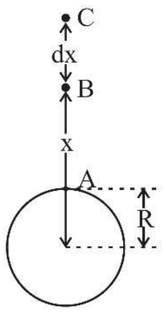
## Escape velocity

It is the minimum velocity with which a body must be thrown upwards from the surface of a planet such that it just crosses the gravitational field of planet and never returns on its own.




Consider a body of mass m at a distance $x$ from the centre of earth as shown. Force acting on this body is

$$F = \frac{GMm}{x^2}.$$

Small amount of work (dW) done to move mass m through small distance d$x$ away from earth is

$$dW = Fdx$$

$$\Rightarrow dW = \frac{GMm}{x^2}dx$$

Therefore, total work done to move this body from x = R to $x = \infty$ is given by

$$\Rightarrow W = GMm\left[\frac{x^{-2+1}}{-2+1}\right]_R^\infty$$

$$\Rightarrow W = -GMm\left[\frac{1}{x}\right]_R^\infty$$

$$\Rightarrow W = -GMm\left[\frac{1}{\infty} - \frac{1}{R}\right]$$

$$\Rightarrow W = \frac{GMm}{R}$$

This work must be equal to KE given to the body at the time of launch

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

$$\Rightarrow \boxed{v_e = \sqrt{\frac{2GM}{R}}}$$

## Alternate method

Since total energy of the body at the surface and at infinity must be equal. Therefore, we can write

$$\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R}\right) = 0 + 0$$

$$\Rightarrow \boxed{v_e = \sqrt{\frac{GMm}{R}}}$$

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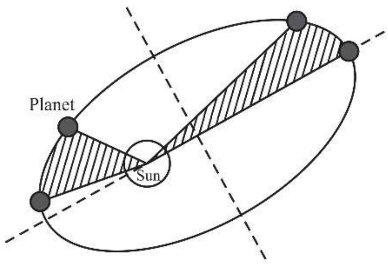
# Kepler's laws of planetary motion

*First law (law of orbits):* Orbits of planets are elliptical in shape and sun is situated at one of the foci.




*Second law (Law of areas):* The area swept by planet per unit time with respect to sun i.e. areal velocity of a planet around sun is always constant.

*Third law (law of time periods):* The square of time period of a planet around sun is directly proportional to the cube of its average orbital radius.

Proof:

Let mass of planet be m and mass of sun be M and average orbital radius of planet is r, then

Centripetal force = gravitational force

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow v^2 = \frac{GM}{r}$$

$$\Rightarrow \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

$$\Rightarrow \frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$\Rightarrow T^2 = \frac{4\pi^2}{GM} r^3$$

Since $\frac{4\pi^2}{GM}$ is constant, therefore $T^2 \propto r^3$

For two planets having time periods $T_1$ and $T_2$

and orbital radii $R_1$ and $R_2$, we can write

$$\boxed{\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}}$$

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