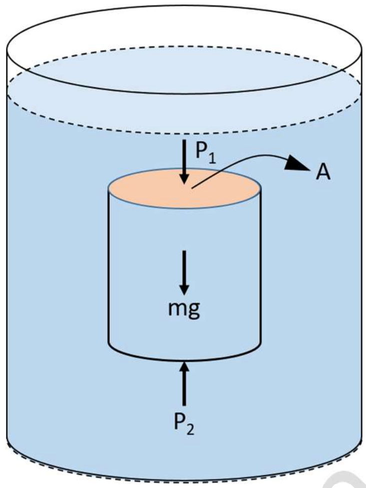
MECHANICAL PROPERTIES OF FLUIDS: ALL DERIVATIONS

Variation of pressure with depth

Imagine a cylindrical element of the liquid of cross-sectional area A and height h. Let P_1 and P_2 be the liquid pressures at its top point 1 and bottom point 2 respectively.



Various force acting on it in the vertical direction are:

1. Force due to the liquid pressure at the top,
   $F_1 = P_1 A$, acting downwards
2. Force due to the liquid pressure at the bottom,
   $F_2 = P_2 A$, acting upwards
3. Weight of the liquid cylinder acting downwards,
   $W = \text{Mass} \times g = \text{Volume} \times \text{density} \times g$

   $= Ah\rho g$

   where p is the density of the liquid.

As the liquid cylinder is in equilibrium,

Net downward force = Net upward force

or $F_1 + W = F_2$

or $F_2 - F_1 = W$

or $P_2 A - P_1 A = Ah\rho g$

or $P_2 - P_1 = h\rho g$

If we shift the point 1 to the liquid surface, which is open to the atmosphere, then we can replace P1 by atmospheric pressure P_a and P_2 by P in the above equation and we get

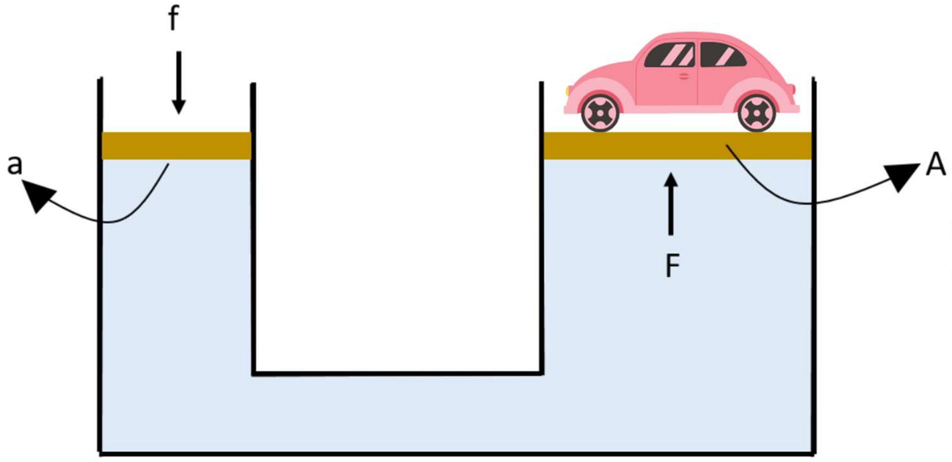
$$P - P_a = h\rho g$$

$$P = P_a + h\rho g$$

The excess pressure $P - P_a$ at depth h is called a **gauge pressure** at that point.

Hydraulic lift

Hydraulic lift is used to lift heavy objects. It works on the principle of Pascal's law.



It has a chamber having two openings as shown. This chamber contains an ideal fluid. This chamber has two openings fitted with piston one having smaller area of cross section and other having larger area of cross section. The object to be lifted is placed on piston with large area of cross-section and a force is applied on the piston with smaller area of cross-section.

Suppose a force f be applied on smaller piston having area of cross-section a. Due to this pressure changes at this cross-section and according to Pascal's law, this pressure gets transmitted equally to piston with larger area of cross-section. If A is the area of cross-section of larger piston and F be the force exerted by liquid on this piston, then according to Pascal's law

$$\frac{f}{a} = \frac{F}{A}$$

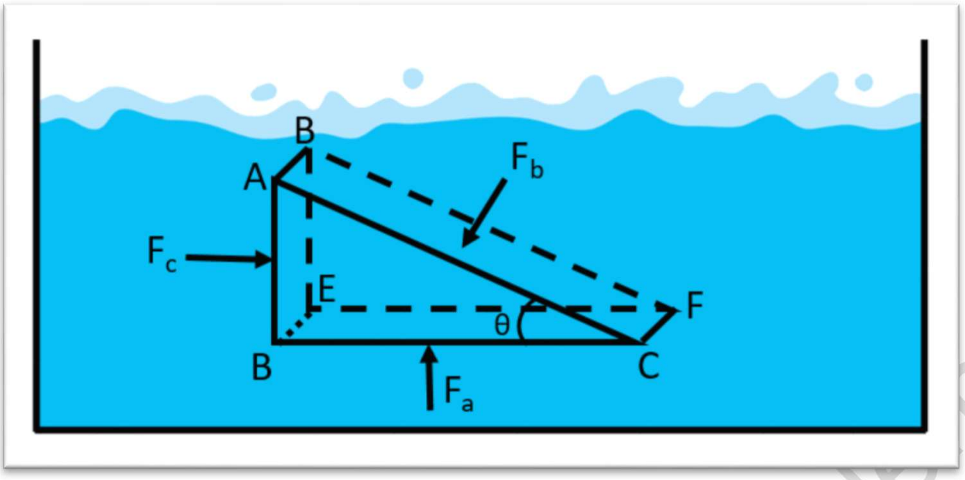
$\because a << A$

$\therefore F >> f$

This force is enough to lift the heavy object.

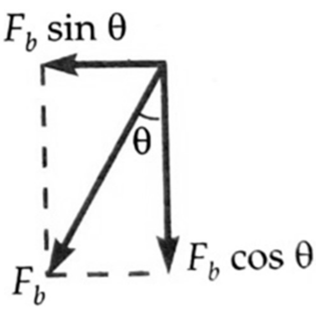
Pascal's law

According to Pascal's law a change in pressure applied to an enclosed incompressible fluid is transmitted undiminished to every point of the fluid and the walls of the containing vessel.



As shown in figure, consider a small element ABC-DEF in the form of a right angled prism in the interior of a fluid at rest. The element is so small that all its parts can be assumed to be at same depth from the liquid surface and, therefore, the effect of gravity is same for all of its points.

Suppose the fluid exerts pressures Pa, Pb and Pc on the faces BEFC, ADFC and ADEB respectively of this element and corresponding normal forces on these faces are Fa, Fb and Fc. Let Aa, Ab and Ac be the areas of the three faces. In right ΔABC, let $\angle ACB = \theta$.



As this element is in equilibrium with remaining fluid, the forces should balance in all directions.

Along horizontal direction

$$F_b \sin\theta = F_c$$

$$F_b \cos\theta = F_a$$

From the geometry of the figure, we get

$$A_b \sin\theta = A_c$$

$$A_b \cos\theta = A_a$$

From above equations, we get

$$\frac{F_b \sin\theta}{A_b \sin\theta} = \frac{F_c}{A_c}$$

and

$$\frac{F_b \cos\theta}{A_b \cos\theta} = \frac{F_a}{A_a}$$

$$\therefore \frac{F_a}{A_a} = \frac{F_b}{A_c} = \frac{F_c}{A_c}$$

or $P_a = P_b = P_c$

Stoke's law

According to Stoke's law, the backward viscous force acting on a small spherical body of radius r moving with uniform velocity v through fluid of viscosity η is given by

$$\boxed{F = 6\pi\eta rv}$$

Derivation of Stoke's law

The viscous force F acting on a sphere moving through a fluid may depend on

i. Coefficient of viscosity η of the fluid
ii. Radius r of the spherical body
iii. Velocity v of the body

Let

$$F = k\eta^a r^b v^c \qquad \text{........(i)}$$

where k is a dimensionless constant

$$[F] = [MLT^{-2}],\ \eta = [ML^{-1}T^{-1}]$$
$$[r] = [L], \qquad v = [LT^{-1}]$$

Substituting these dimensions in eq. (i), we get

$$[MLT^{-2}] = [ML^{-1}T^{-1}]^a [L]^b [LT^{-1}]^c$$

Equating the powers M, L and T on both sides, we get

$$a = 1$$
$$-a + b + c = 1$$
$$-a - c = -2$$

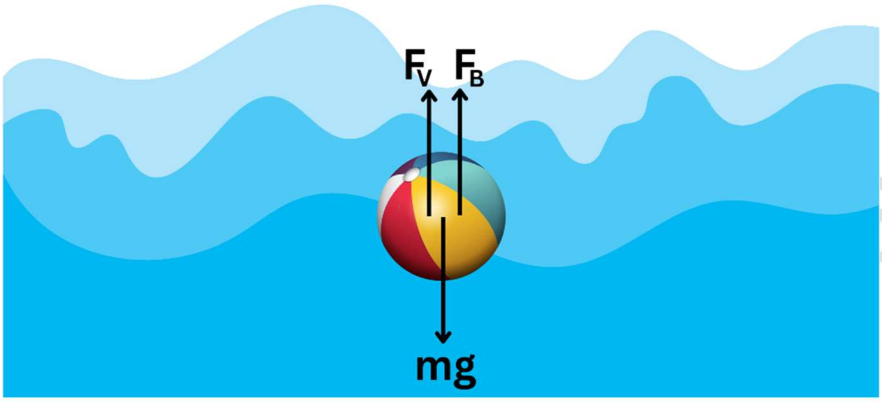
On solving we get a = b = c = 1

For a small sphere k is found to be 6π

Hence $F = 6\pi\eta rv$

Terminal Velocity

Terminal Velocity. The maximum constant velocity acquired by a body while falling through a viscous medium is called as Terminal Velocity.



Expression for terminal velocity. Consider a spherical body of radius r falling through a viscous liquid of density of the body.

As the body falls, the various forces acting on the body are:

1. Weight of the body acting vertically downwards.

$$W = mg = \frac{4}{3}\pi r^3 \rho g$$

2. Upward thrust equal to the weight of the liquid displaced.

$$F_B = \frac{4}{3}\pi r^3 \sigma g$$

3. Force of viscosity acting in the upward direction. According to Stoke's Law,

$$F_v = 6\pi\eta r v$$

When the body attains terminal velocity v ,

$$F_V + F_B = W$$

$$\Rightarrow \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v_t = \frac{4}{3}\pi r^3 \rho g$$

$$\Rightarrow 6\pi\eta r v_t = \frac{4}{3}\pi r^3 (\rho - \sigma) g$$

Or

$$\boxed{v_t = \frac{2r^2(\rho - \sigma)}{9\eta} g}$$

This is the expression for terminal velocity.

Critical velocity

"The critical velocity of a liquid is that limiting value of its velocity of flow up-to which the flow is streamlined and above which the flow becomes turbulent."

The critical velocity v_c of a liquid flowing through a tube depends upon

i. Coefficient of viscosity of liquid (η)
ii. Density of liquid (ρ)
iii. Diameter of the tube (D)

Let $v_c = k\eta^a \rho^b D^c$

Where k is a dimensionless constant. Writing the above equation in dimension form, we get

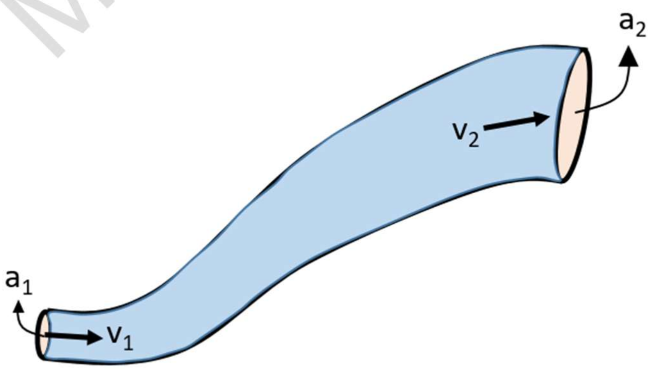
$$[M^0LT^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c$$
$$[M^0LT^{-1}] = [M^{a+b}L^{-a-3b+c}T^{-a}]$$

Equating powers of M, L and T we get

$$a + b = 0$$
$$-a - 3b + c = 1$$
$$-a = -1$$
$$\therefore a = 1, b = -1, c = -1$$
$$\therefore v_c = k\eta\rho^{-1}D^{-1}$$
$$\Rightarrow \boxed{v_c = \frac{k\eta}{\rho D}}$$

Equation of continuity

Consider a non-viscous and incompressible liquid flowing steadily between the sections A and B of a pipe of varying cross section, Let a_1 be the area of cross section, v_1 fluid velocity, ρ_1 fluid density at section A; and the values of corresponding quantities at section B be a_2, v_2 and ρ_2.



As m = volume x density

= area of cross section x length x density

Therefore, mass of fluid that flows through section A in time Δt,

$m_1 = a_1 v_1 \Delta t \rho_1$

Mass of fluid that flows through section B in time Δt,

$m_2 = a_2 v_2 \Delta t \rho_2$

By conservation of mass

$m_1 = m_2$

$a_1 v_1 \Delta t \rho_1 = a_2 v_2 \Delta t \rho_2$

As fluid is incompressible, $\rho_1 = \rho_2$ and hence

$\boxed{a_1 v_1 = a_2 v_2}$

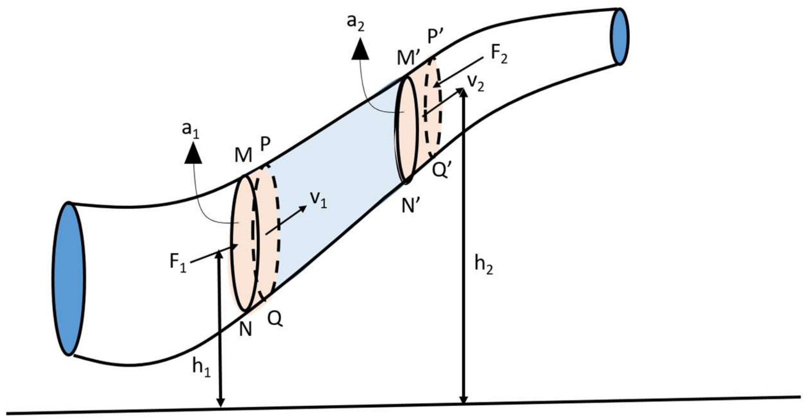
Bernoulli's Principle

Bernoulli's Principle states that the sum of pressure energy, kinetic energy and potential energy per unit volume of an incompressible, non-viscous fluid in a streamlined irrotational flow remains constant along a streamline.

Mathematically, it can be expressed as

$$\boxed{P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$$

Proof. Consider a non-viscous and incompressible fluid flowing steadily flowing through a pipe of varying cross-section. Let a1 be the area of cross-section at MN, v_1 the fluid velocity, P_1 the fluid pressure, and h_1 the mean height above the ground level. Let a_2, v_2, P_2 and h_2 be the values of the corresponding quantities at M'N'.



Let ρ be the density of the fluid. Let the part MNM'N' of the liquid moves to PQP'Q' in time Δt

As liquid is incompressible so mass of liquid in MNPQ part and M'N'P'Q' is same which is given by

m = Volume x Density = Area of cross-section x length x density

or $m = a_1 v_1 \Delta t p = a_2 v_2 \Delta t p$

K.E of the fluid = K.E at B – K.E at A

or $a_1 v_1 = a_2 v_2$

$\therefore$ Change in K.E

$$= \frac{1}{2} m\left(v_2^2 - v_1^2\right) = \frac{1}{2} a_1 v_1 \Delta t \rho\left(v_2^2 - v_1^2\right)$$

Change in P.E of the fluid

= P.E at B – P.E at A

$$= mg\left(h_2 - h_1\right) = a_1 v_1 \Delta t \rho g\left(v_2^2 - v_1^2\right)$$

Net work done on the fluid

= work done on the fluid A – Work done by the fluid at B

$$= P_1 a_1 v_1 \Delta t - P_2 a_2 v_2 \Delta t$$
$$= a_1 v_1 \left(P_1 - P_2\right)$$

By conservation of energy,

Net work done on the fluid

= change in K.E of the fluid + change in P.E of the fluid

$$\therefore a_1 v_1 \Delta t\left(P_1 - P_2\right) = \frac{1}{2} a_1 v_1 \Delta t \rho\left(v_2^2 - v_1^2\right) + a_1 v_1 \Delta t \rho g\left(h_2 - h_1\right)$$

Dividing both sides by $a_1 v_1 \Delta t$, we get

$$P_2 - P_1 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 + \rho g h_2 - \rho g h_1$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho h g_2$$

or $\boxed{P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$

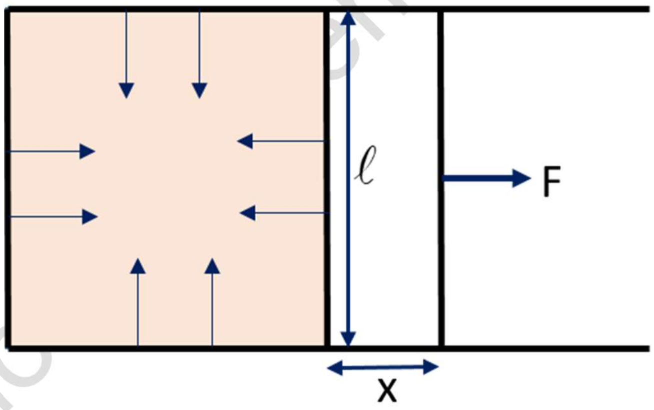
Surface energy

Work done to increase the area of a liquid surface against the force of surface tension gets stored in the surface in the form of surface energy.

Consider a metallic frame with a movable arm of length ℓ. Now, this frame is dipped in a liquid solution so a soap film is formed on it as shown. Now, we are applying a force F against the force of surface tension and moving the arm through a distance x, so work done is

$$W = F \times x$$

Since $S = \frac{F}{2\ell} \Rightarrow F = S \times 2\ell$



Therefore

$$W = S \times 2\ell \times x$$
$$\Rightarrow W = S \times \Delta A$$

This work is stored in the liquid surface in the form of surface energy.

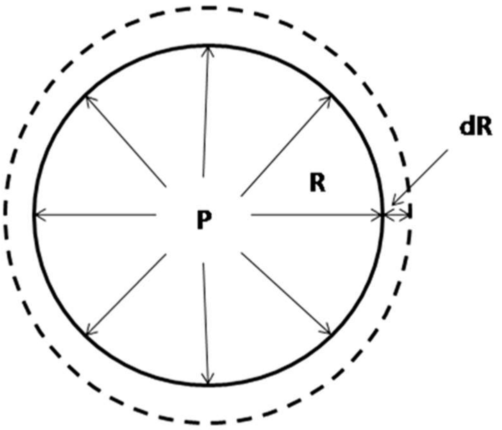
Hence
$$\text{Surface energy} = \text{Surface tension} \times \text{increase in area}$$

Excess pressure inside a liquid drop

Consider a spherical liquid drop of radius R. Let σ be the surface tension of the liquid. Due to its spherical shape, there is an excess pressure p inside the drop over that on outside. This excess pressure acts normally outwards. Let the radius of the drop increase from R to R + dR under the excess pressure p.

Initial surface area = $4\pi R^2$

Final surface area =



$$4\pi(R+dR)^2 = 4\pi(R^2+2RdR + dR^2)$$
$$= 4\pi R^2 + 8\pi R\, dR$$
dR^2 is neglected as it is small.

Increase in surface area

$= 4\pi R^2 + 8\pi R\, dR - 4\pi R^2 = 8\pi R\, dR$

Work done in enlarging the drop

= Increase in surface energy

= Increase in surface area × Surface tension

$= 8\pi R\, dR\sigma$

But work done = Force × Distance

= Pressure × Area × Distance

$= p \times 4\pi R^2 \times dR$

Hence, $p \times 4\pi R^2 \times dR = 8\pi R\, dR\sigma$

Excess Pressure,

$$\boxed{p = \frac{2\sigma}{R}}$$

Excess pressure inside a soap bubble

Proceeding as in the case of a liquid drop in the above derivation, we obtain

Increase in surface area = $8\pi R\, dR$

But a soap bubble has air both inside and outside, so it has two free surfaces

∴ Effective increase in surface area

$= 2 \times 8\pi R\, dR = 16\pi R\, dR$

Work done in enlarging the soap bubble

= Increase in surface energy

= Increase in surface area x Surface tension

$= 16\pi R\, dR\, \sigma$

But, Work done = Force x Distance

$= p \times 4\pi R^2 \times dR$

Hence

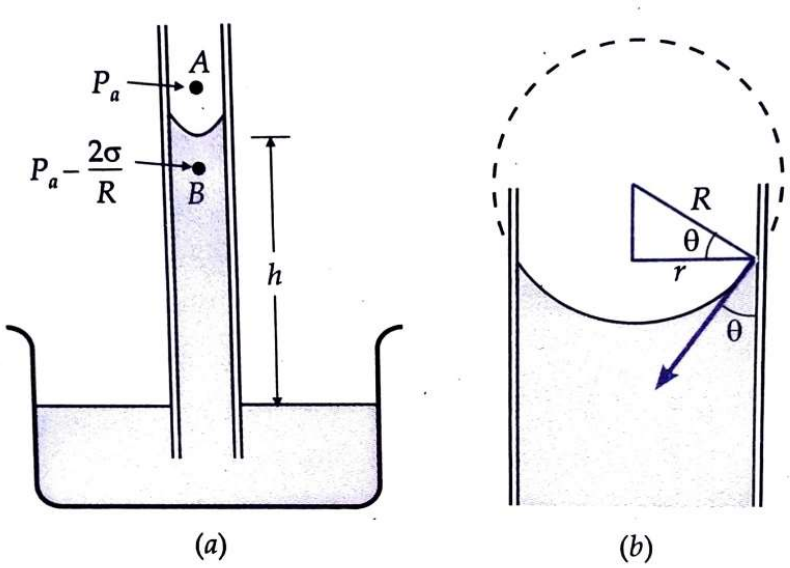
$$p \times 4\pi R^2 \times dR = 16\pi R\, dR\, \sigma$$

or $$\boxed{p = \frac{4\sigma}{R}}$$

Ascent formula

Consider a capillary tube of radius r dipped in a liquid of surface tension S and density ρ. Suppose the liquid wets the sides of the tube. Then its meniscus will be concave. The shape of the meniscus of water will be nearly spherical if the capillary tube is of sufficiently narrow bore.

As the pressure is greater on the concave side of a liquid surface, so excess of pressure at a point A just above the meniscus compared to point B just below the meniscus



$$p = \frac{2S}{R}$$

Where R is the radius of curvature of meniscus. If θ is the angle of contact then from right angled triangle shown in figure, we have

$$\frac{r}{R} = \cos\theta$$

or $$R = \frac{r}{\cos\theta}$$

$$\Rightarrow \quad p = \frac{2S\cos\theta}{r}$$

Due this excess pressure p, the liquid rises in the capillary tube t height h when the hydrostatic pressure exerted by the liquid column becomes equal to the excess pressure p. Therefore, at equilibrium, we have

$$p = h\rho g$$

or $$\frac{2S\cos\theta}{r} = h\rho g$$

or $$\boxed{p = \frac{2S\cos\theta}{r\rho g}}$$