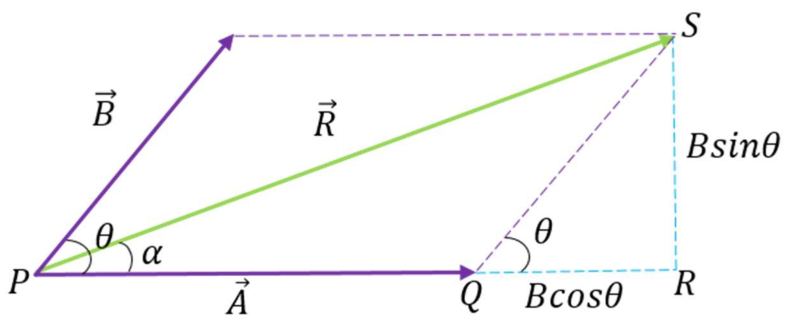
Motion in a plane – all derivations

Parallelogram law of vector addition

According to the parallelogram law of vector addition: If two vectors are considered to be the adjacent sides of a parallelogram, then the resultant of the two vectors is given by the vector that is diagonal passing through the point of contact of the two vectors.

Consider two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ as shown. Let their resultant be $\vec{R}$.



Extend PQ and draw QR such that $SR \perp QR$.

In ΔQRS

$$\frac{QR}{QS} = \cos\theta \Rightarrow QR = QS\cos\theta = B\cos\theta \qquad ....(i)$$

$$\frac{SR}{QS} = \sin\theta \Rightarrow SR = QS\sin\theta = B\sin\theta \qquad ....(ii)$$

In ΔPSR

$$(PS)^2 = (PR)^2 + (SR)^2$$

$$\Rightarrow R^2 = (PQ + QR)^2 + (SR)^2$$

$$\Rightarrow R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2 \qquad \text{[using (i) and (ii)]}$$

$$\Rightarrow R^2 = A^2 + B^2\sin^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow R^2 = A^2 + B^2(\sin^2\theta + \cos^2\theta) + 2AB\cos\theta$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow \boxed{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then

$$\tan\alpha = \frac{SR}{PR} = \frac{SR}{PQ+QR}$$

$$\Rightarrow \tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}$$

$$\Rightarrow \boxed{\alpha = \tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)}$$

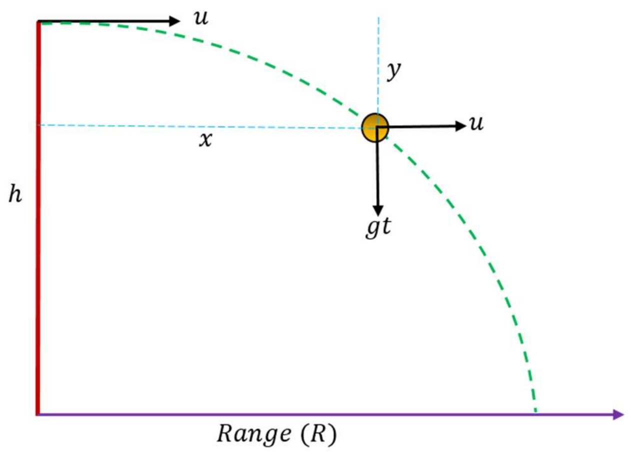
Horizontal projection of projectile

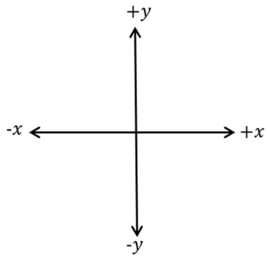
Consider a projectile thrown with velocity u in horizontal direction from a height h as shown

Therefore,

$u_x = u, \quad u_y = 0$

$a_x = 0, \quad a_y = -g$





Equation of path

$$x = u_x t + \frac{1}{2}a_x t^2 \Rightarrow \boxed{x = ut}$$

$$y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow \boxed{y = -\frac{1}{2}gt^2}$$

$$\because t = \frac{x}{u}$$

$$y = -\frac{1}{2}g\left(\frac{x}{u}\right)^2 = -\frac{1}{2}\frac{g}{u}x^2$$

Which is a quadratic equation. Thus, path of a projectile is parabolic in nature.

Time of flight

Total time for which the projectile remains in air is called time of flight.

$$\because y = u_y t + \frac{1}{2} a_y t^2$$

$$\therefore -h = (0)t - \frac{1}{2} g T^2$$

$$\Rightarrow \boxed{T = \sqrt{\frac{2h}{g}}}$$

Horizontal range (R)

Maximum horizontal distance travelled by projectile.

$$\because x = u_x t + \frac{1}{2} a_x t^2$$

$$\therefore R = uT + \frac{1}{2}(0)T^2$$

$$\Rightarrow \boxed{R = u\sqrt{\frac{2h}{g}}}$$

$$\therefore -h = (0)t - \frac{1}{2} g T$$

Velocity at any instant

$$v_x = u_x + a_x t$$

$$\Rightarrow \boxed{v_x = u}$$

$$v_y = u_y + a_y t$$

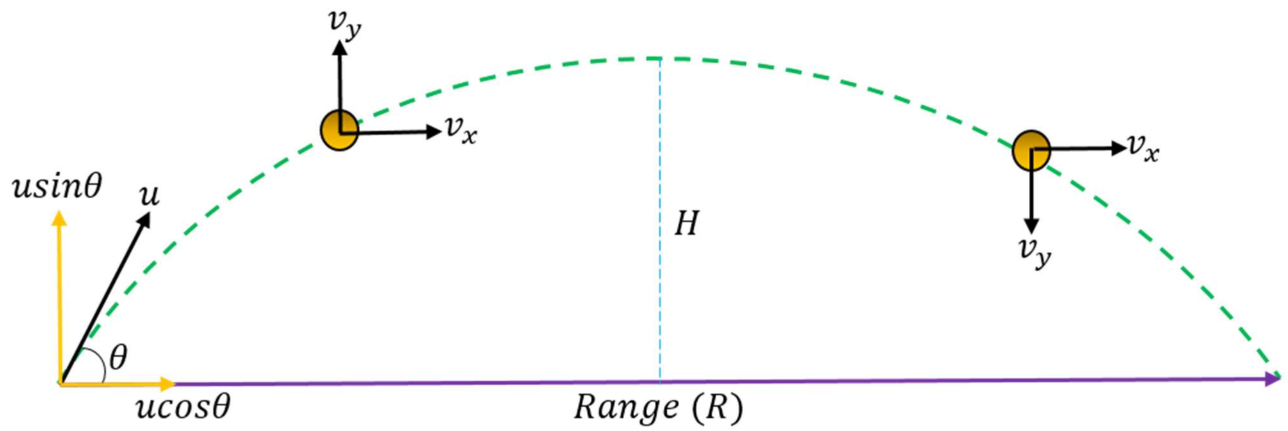
$$\Rightarrow \boxed{v_y = -gt}$$

$$\text{As } v = \sqrt{v_x^2 + v_y^2}$$

$$\boxed{v = \sqrt{u^2 + g^2 t^2}}$$

Angular projectile motion

Consider a body projected with velocity u at an angle θ with horizontal as shown.



Therefore,

$u_x = u\cos\theta,\ \ u_y = u\sin\theta$

$a_x = 0 \qquad a_y = -g$

Equation of path

$$x = u_x t + \frac{1}{2}a_x t^2 \Rightarrow \boxed{x = u\cos\theta t}$$

$$y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow \boxed{y = u\sin\theta t - \frac{1}{2}gt^2}$$

$$\because t = \frac{x}{u\cos\theta}$$

$$\therefore y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\Rightarrow \boxed{y = x\tan\theta - \frac{1}{2}\left(\frac{g}{u^2\cos^2\theta}\right)x^2}$$

Time of flight

Total time for which the projectile remains in air is called time of flight.

$$\because y = u_y t + \frac{1}{2}a_y t^2$$

$$\therefore 0 = (u\sin\theta)T - \frac{1}{2}gT^2 \quad [y = 0 \text{ when body hits the ground}]$$

$$\Rightarrow \boxed{T = \frac{2u\sin\theta}{g}}$$

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Maximum height attained

At maximum height $v_y = 0$

$$\therefore 0 = u_y + a_y t$$
$$\Rightarrow 0 = u\sin\theta - gt$$
$$\Rightarrow t = \frac{u\sin\theta}{t}$$

Putting this value in equation of y, we get

$$y = u_y t + \frac{1}{2} a_y t^2$$
$$\Rightarrow H = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2} g\left(\frac{u\sin\theta}{g}\right)^2$$
$$\Rightarrow H = \frac{u^2 \sin^2\theta}{g} - \frac{1}{2}\frac{u^2 \sin^2\theta}{g}$$
$$\Rightarrow \boxed{H = \frac{u^2 \sin^2\theta}{2g}}$$

Horizontal range

$$\because x = u_x t + \frac{1}{2} a_x t^2$$
$$R = u\cos\theta\left(\frac{2u\sin\theta}{g}\right) + \frac{1}{2}(0)t^2$$
$$\Rightarrow R = \frac{2u^2 \sin\theta\cos\theta}{g}$$
$$\Rightarrow \boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

Velocity at any instant

$$v_x = u_x + a_x t = u\cos\theta$$
$$v_y = u_y + a_y t = u\sin\theta - gt$$
$$\because v = \sqrt{v_x^2 + v_y^2}$$

$$\Rightarrow v = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$$

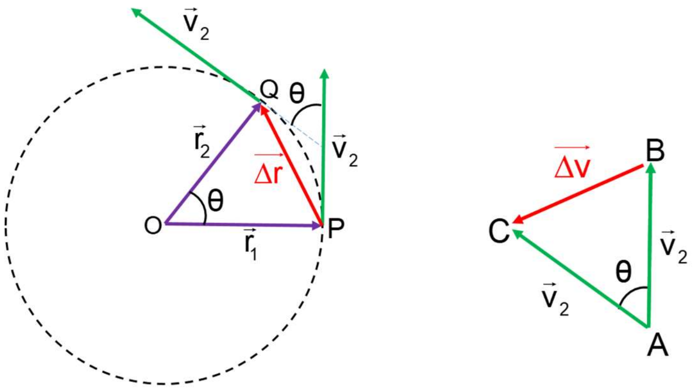
$$\Rightarrow v = \sqrt{u^2\sin^2\theta + u^2\cos^2\theta + g^2t^2 - 2u\sin\theta gt}$$

$$\Rightarrow v = \sqrt{u^2\left(\sin^2\theta + \cos^2\theta\right) + g^2t^2 - 2u\sin\theta gt}$$

$$\Rightarrow \boxed{v = \sqrt{u^2 + g^2t^2 - 2u\sin\theta gt}}$$

Centripetal acceleration

Consider a body moving in a circle of radius r with velocity v. Let the position vector of body be $\vec{r}_1$ when it is at P and $\vec{r}_2$ when it is Q. The velocity vector of body at P is $\vec{v}_1$ and Q it is $\vec{v}_2$. If angle between $\vec{r}_1$ and $\vec{r}_2$ is θ then clearly angle between $\vec{v}_1$ and $\vec{v}_2$ is also θ.



Clearly $|\vec{r}_1| = |\vec{r}_2| = r$

Since the motion is uniform so, $|\vec{v}_1| = |\vec{v}_2| = v$

Now in

ΔQOP and ΔCAB

$$\frac{OP}{AB} = \frac{OQ}{AC} = \frac{r}{v}$$

and $\angle QOP = \angle CAB$

$\therefore$ by SAS similarity rule

$\Delta QOP \sim \Delta CAB$

so, $\frac{\Delta v}{\Delta r} = \frac{v}{r} \Rightarrow \frac{\Delta v}{\Delta t} = \frac{v}{r}\left(\frac{\Delta r}{\Delta t}\right)$

$$\Rightarrow \boxed{a_c = \frac{v^2}{r}}$$

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