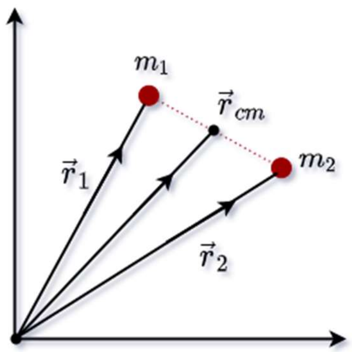
SYSTEM OF PARTICLES – ALL DERIVATIONS

CENTER OF MASS OF A TWO PARTICLE SYSTEM

Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r}_1$ and $\vec{r}_2$ be their position vectors at any instant t with respect to the origin O, as shown in Fig.



Let m_1, m_2 be masses of two particles

let r_1, r_2 be position vectors of particles

let f_1, f_2 be external forces on particles

Let v_1, v_2 be velocities of particles

Let F_{12}, F_{21} be internal forces on particles (due to each other)

According's to Newton's second law:

$$\vec{f}_1 + \vec{f}_2 + \vec{F}_{12} + \vec{F}_{21} = \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2)$$

As $\vec{v}_1 = \frac{\overrightarrow{dr_1}}{dt}$ and $\vec{v}_2 = \frac{\overrightarrow{dr_2}}{dt}$

$$\vec{f}_1 + \vec{f}_2 + \vec{F}_{12} + \vec{F}_{21} = \frac{d}{dt}\left(m_1\frac{\overrightarrow{dr_1}}{dt} + m_2\frac{\overrightarrow{dr_2}}{dt}\right)$$

But $\vec{F}_{12} = -\vec{F}_{21}$ so they will cancel out

$$\vec{f}_1 + \vec{f}_2 = \frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2)$$

Multiply and divide L.H.S by $m_1 + m_2$ we get

$$\vec{f}_1 + \vec{f}_2 = (m_1 + m_2)\frac{d^2}{dt^2}\frac{(m_1\vec{r}_1 + m_2\vec{r}_2)}{(m_1 + m_2)}$$

Let $f = \vec{f}_1 + \vec{f}_2$

$$\vec{f} = (m_1 + m_2)\frac{d^2}{dt^2}\frac{(m_1\vec{r}_1 + m_2\vec{r}_2)}{(m_1 + m_2)}$$

Comparing this equation with $\vec{f} = (m_1 + m_2)\frac{d^2}{dt^2}\vec{R}_{cm}$, we get

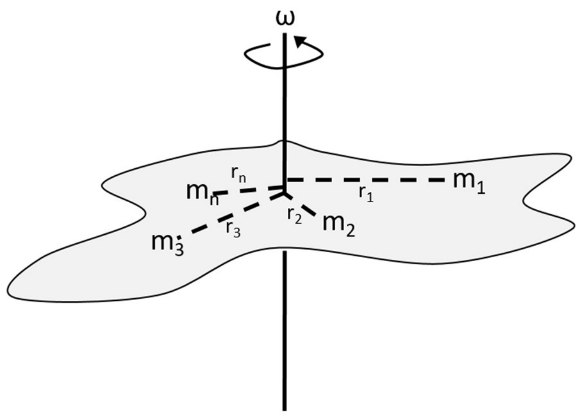
$$\boxed{\vec{R}_{cm} = \frac{(m_1\vec{r}_1 + m_2\vec{r}_2)}{(m_1 + m_2)}}$$

Centre of mass of an n particle system

$$\boxed{\vec{R}_{cm} = \frac{(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 .... + m_n\vec{r}_n)}{(m_1 + m_2 + m_3 + ...m_n)}}$$

RELATION BETWEEN ROTATIONAL K.E. AND MOMENT OF INERTIA

Relation between rotational kinetic energy and moment of inertia. As shown in Fig., consider a rigid body rotating about an axis with uniform angular velocity ω. The body may be assumed to consist of n particles of masses m_1 , m_2 , m_3 ,..........m_n ; situated at distances r_1 , r_2 , r_3 ,..........r_n from the axis of rotation. As the angular velocity ω of all the n particles is same, so their linear velocities are



$v_1 = r_1\omega$, $v_2 = r_2\omega$, $v_3 = r_3\omega$,.........,$v_n = r_n\omega$

Hence the total kinetic energy of rotation of the body about the axis of rotation is Rotational K.E.

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ... + \frac{1}{2}m_nv_n^2$$

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + ...\frac{1}{2}m_nr_n^2\omega^2$$

$$= \frac{1}{2}\left(m_1r_1^2 + m_2r_2^2 + ...m_nr_n^2\right)\omega^2$$

$$= \frac{1}{2}I\omega^2$$

where $I = m_1r_1^2 + m_2r_2^2 + ...m_nr_n^2$ (moment of inertia)

RADIUS OF GYRATION

The radius of gyration of a body about it axis of rotation may be defined as the distance from the axis of rotation at which, if the whole mass of the body were concentrated, its moment of inertia about the given axis would be the same as with the actual distribution of mass.

Expression for k. Suppose a rigid body consists of n particles of mass m each, situated at distances r_1 , r_2 , r_3 ,..........r_n from the axis of rotation AB.

The moment of inertia of the body about the axis AB is

$$I = mr_1^2 + mr_2^2 + mr_3^2 + ...m_n r_n^2$$
$$I = m\left(r_1^2 + r_2^2 + r_3^3 + ...\right)$$

Multiplying and dividing RHS by n, we get

$$I = \frac{m \times n\left(r_1^2 + r_2^2 + r_3^3 + ...\right)}{n}$$

now m × n = M, total mass of the body.

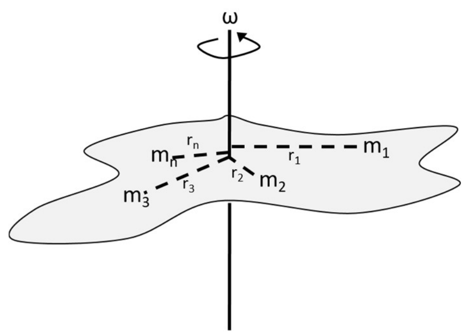
If k is the radius of gyration about the axis AB, then

$I = MK^2$, therefore

$$Mk^2 = \frac{M\left(r_1^2 + r_2^2 + r_3^3 + ...\right)}{n}$$
$$\Rightarrow \boxed{k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^3 + ...}{n}}}$$

RELATION BETWEEN ANGULAR MOMENTUM AND MOMENT OF INERTIA

Consider a rigid body rotating about a fixed axis with uniform angular velocity ω. The body consists of n particles of masses m_1, m_2, m_3, ..., m_n ;situated at distance r_1 , r_2 , r_3 ......, r_n from the axis of rotation. The angular velocity ω of all the n particles will be same but their linear velocities will be different and are given by



$v_1 = r_1\,\omega$, $v_2 = r_2\,\omega$, $v_3 = r_3\,\omega$,......., $v_n = r_n\,\omega$

Linear momenta of particles,

$p_1 = m_1 r_1 = m_1 r_1 \omega$,

$p_2 = m_2r_2 = m_2r_2\omega$,

$p_3 = m_3r_3 = m_3r_3\omega$…………

Therefore, the angular momenta of particles

$L_1 = p_1r_1 = m_1r_1^2\omega$

$L_2 = p_2r_2 = m_2r_2^2\omega$

$L_3 = p_3r_3 = m_3r_2^2\omega$ ………

The angular momentum of a rigid body about an axis is the sum of moments of linear momenta of all its particles about that axis. Thus

$$L = L_1 + L_2 + L_3 + ..., + L_n$$

$$= m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + ...m_nr_n^2\omega$$

$$= \left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...m_nr_n^2\right)\omega$$

Since

$$m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...m_nr_n^2 = I \text{ (moment of inertia)}$$

$$\therefore L = I\omega$$

RELATION BETWEEN ANGULAR MOMENTUM AND TORQUE

As Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

and Angular Momentum, $\vec{L} = \vec{r} \times \vec{p}$

Differentiating both sides w.r.t. time t, we get

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\Rightarrow \vec{v} \times \vec{p} + \vec{r} \times \vec{F} \qquad \left[\because \frac{d\vec{p}}{dt} = \vec{F}\right]$$

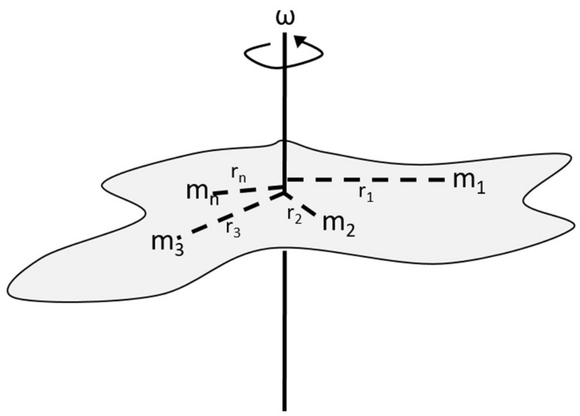
$$\Rightarrow 0 + \vec{\tau} \qquad \left[\because \vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = 0\right]$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

RELATION BETWEEN TORQUE AND MOMENT OF INERTIA

Consider a particle P of mass m_1 at a distance r_1 from the axis of rotation. Let its linear velocity be v_1 .

Linear acceleration of the first particle, $a_1 = r_1 \alpha$



Moment of force F_1 about the axis rotation is

$$\tau_1 = r_1 f_1 = m_1 r_1^2 \alpha$$

Similarly, $\tau_2 = r_2 f_2,\ \tau_3 = r_3 f_3, \ldots\ldots\ldots\ldots\ldots \tau_n = r_n f_n,$

Total torque acting on the rigid body is

$$\tau = \tau_1 + \tau_2 + \tau_3 + \ldots\ldots\ldots + \tau_n$$

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \ldots\ldots . m_n r_n^2 \alpha$$

$$\Rightarrow \tau = \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots . m_n r_n^2\right)\alpha$$

Since $m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots . m_n r_n^2 = I$ (moment of inertia)

$$\therefore \boxed{\tau = I\alpha}$$

EQUATIONS OF ROTATIONAL MOTION

First equation of motion. Consider a rigid body rotating about a fixed axis with constant angular acceleration α. By definition,

$$\alpha = \frac{d\omega}{dt}$$

$$\therefore d\omega = \alpha dt$$

Integrting both sides within limits

$$\int_{\omega_1}^{\omega_2} d\omega = \alpha \int_0^t dt$$

$$\Rightarrow [d\omega]_{\omega_1}^{\omega_2} = \alpha[t]_0^t$$

$$\Rightarrow \omega_2 - \omega_1 = \alpha[t - 0]$$

$$\Rightarrow \boxed{\omega_2 = \omega_1 + \alpha t}$$

Second equation of motion. Let ω_2 be the angular velocity of a rigid body at any instant t. By definition,

$$\omega_2 = \frac{d\theta}{dt}$$

$$d\theta = \omega_2 dt$$

$$\Rightarrow d\theta = (\omega_1 + \alpha t)dt$$

Integrting both sides within limits

$$\int_0^\theta d\theta = \omega_1 \int_0^t dt + \alpha \int_0^t t dt$$

$$(\theta)_0^\theta = \omega_1 (t)_0^t + \alpha \left(\frac{t^2}{2} \right)_0^t$$

$$(\theta - 0) = \omega_1 (t - 0) + \alpha \left(\frac{t^2}{2} - 0 \right)$$

$$\boxed{\theta = \omega_1 t + \frac{1}{2} \alpha t^2}$$

Third equation of motion. The angular acceleration α may be expressed as

$$\alpha = \frac{d\omega}{dt}$$

multiply and divide by $d\theta$, we get

$$\alpha = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt}$$

$$\alpha = \omega \frac{d\omega}{d\theta}$$

$$\Rightarrow \alpha d\theta = \omega d\omega$$

Integrating both sides within given limits

$$\alpha \int_0^\theta d\theta = \int_{\omega_1}^{\omega_2} \omega d\omega$$

$$\Rightarrow \alpha (\theta)_0^\theta = \left(\frac{\omega^2}{2} \right)_{\omega_1}^{\omega_2}$$

$$\Rightarrow \alpha (\theta - 0) = \left(\frac{\omega_2^2}{2} - \frac{\omega_1^2}{2} \right)$$

$$\Rightarrow \boxed{2\alpha\theta = \omega_2^2 - \omega_1^2}$$

ACCELERATION OF A BODY MOVING DOWN A ROUGH INCLINED PLACE

Consider a body of mass M and radius R rolling down a plane inclined at an angle θ to the horizontal.

The external forces acting on the body are

I. The weight Mg of the body acting vertically downwards through the center of mass of the cylinder.
II. The normal reaction N of the inclined plane acting perpendicular to the plane at P.
III. The frictional force f acting upwards and parallel to the inclined plane.

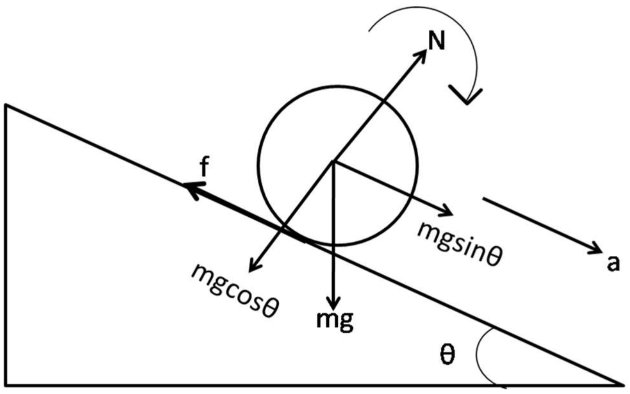
The weight Mg can be resolved into two rectangular components:

I. Mg cos θ perpendicular to the inclined plane.
II. Mg sin θ acting down the inclined plane.

As there is no motion in a direction normal to the inclined plane, so

$$N = Mg \cos \theta$$

Applying Newton's second law to the linear motion of the center of mass, the net force on



the body rolling down the inclined plane is

$$F = Ma = Mg \sin \theta - f$$

It is only the force of friction f which exerts torque τ on the cylinder and makes it rotate with angular acceleration α. It acts tangentially at point of contact P and has lever arm equal to R.

$$\tau = \text{Force x force arm} = fR$$

Also, $\tau = I\alpha$

$$FR = I\alpha$$

Or $$f = \frac{I\alpha}{R}$$

Putting the value of f in equation, we get

$$Mg\sin\theta - f = Ma$$

$$\Rightarrow Mg\sin\theta - \frac{I\alpha}{R} = Ma$$

As $\alpha = \frac{a}{R}$

$$\therefore Mg\sin\theta - \frac{Ia}{R^2} = Ma$$

$$Mg\sin\theta = \frac{Ia}{R^2} + Ma$$

$$\Rightarrow a\left(\frac{I}{R^2} + M\right) = Mg\sin\theta$$

$$\Rightarrow \boxed{a = \frac{Mg\sin\theta}{\left(\frac{I}{R^2} + M\right)}}$$

LAW OF CONSERVATION OF ANGULAR MOMENTUM

It states that if external torque acting on a system is zero, the total angular momentum of the system remains conserved.

Proof:

As we know

$$\tau_{ext} = \frac{dL}{dt}$$

if $\tau_{ext} = 0$

then $\frac{dL}{dt} = 0$

or $L = \text{constant}$