Thermodynamics: All derivations

Question: Derive a relation between two principles specific heats of a gas

Or derive a relation between C_P and C_V

Or derive Mayer's formula.

Consider n moles of an ideal gas. Heat the gas to raise their temperature by dT. According to the first law of thermodynamics, the heat supplied dQ is used to partly to increase the internal energy and partly in doing the work of expansion. That is,

$$dQ = dU + PdV$$

If the heat dQ is absorbed at constant volume, then dV = 0 and we have

$$dQ = nC_V dt \text{ and } dQ = dU$$
$$\therefore dU = nC_V dt \qquad \text{................(i)}$$

If now the heat dQ is absorbed at constant pressure, then

$$dQ = dU + PdV$$
$$\Rightarrow nC_P dt = dU + PdV$$

Change in internal energy is same in both case because temperature change is same.

Using (i) we get

$$nC_P dt = nC_V dt + P\Delta V$$
$$\Rightarrow n\left(C_P - C_V\right)dt = PdV$$

$$\because PV = nRT$$
$$\therefore PdV = nRdT$$

Putting this in above relation, we get

$$n\left(C_P - C_V\right)dt = nRdt$$
$$\text{or } \; C_P - C_V = R$$

This is the required relation between C_P and C_V. It is also known as Mayer's Formula.

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<u>Question:</u> Derive an expression for work done by a gas in adiabatic expansion from volume V_1 to V_2.

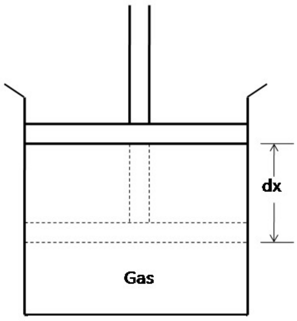
Work done in an adiabatic expansion. Consider n moles of an ideal gas contained in a cylinder having insulating walls and provided with frictionless and insulating piston. Let P be the pressure of the gas. When the piston moves up through a small distance dx, the work done by the gas will be

$$dW = PAdx = p\,dV$$

where A is the cross-sectional area of the piston and dV = Adx is the increase in the volume of the gas.

Suppose the gas expands adiabatically and changes from the initial state (P_1, V_1, T_1) to the final state (P_2, V_2, T_2). The total work done by the gas will be

$$W_{adia} = \int_{V_1}^{V_2} PdV$$



Adiabatic walls

For an adiabatic change $PV^{\gamma} = K$ or $P = KV^{-\gamma}$, $\therefore$

$$W_{adia} = \int_{V_1}^{V_2} KV^{-\gamma}dV$$

$$= K\int_{V_1}^{V_2} V^{-\gamma}dV = K\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_1}^{V_2}$$

$$= \frac{K}{1-\gamma}[V_2^{1-\gamma} - V_1^{1-\gamma}] = \frac{1}{\gamma-1}[KV_1^{1-\gamma} - KV_2^{1-\gamma}]$$

But $K = P_1V_1^{\gamma} = P_2V_2^{\gamma}$

$$W_{adia} = \frac{1}{\gamma-1}[P_1V_1^{\gamma}V_1^{1-\gamma} - P_2V_2^{\gamma}V_2^{1-\gamma}]$$

$$W_{adia} = \frac{1}{\gamma-1}[P_1V_1 - P_2V_2]$$

Also, $P_1V_1 = nRT_1$ and $P_2V_2 = nRT_2$

$$W_{adia} = \frac{nR}{\gamma-1}[T_1-T_2]$$

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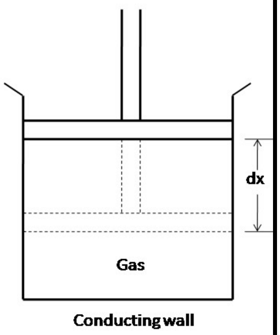
<u>Question:</u> Derive an expression for work done by a gas in an isothermal reversible expansion from volume V_1 to V_2 at constant temperature T.

Work done in an isothermal expansion. Consider n moles of an ideal gas contained in a cylinder having conducting walls and provided with frictionless and movable piston, as shown in the figure below. Let P be the pressure of the gas.



Work done by the gas when the piston moves up through a small distance dx is given by

$$dW = P\,A\,dx = PdV$$

where A is the cross-sectional area of the piston and dV = Adx, is the small increase in the volume of the gas. Suppose the gas expands isothermally from initial state (P_1, V_1) to the final state (P_2, V_2). The total amount of work done will be

$$W_{iso} = \int_{V_1}^{V_2} PdV$$

For n moles of a gas, $PV = nRT$ or $P = \frac{nRT}{V}$

$$\therefore W_{iso} = \int_{V_1}^{V_2} \frac{nRT}{V} dV = nRT \int_{V_1}^{V_2} \frac{1}{V} dV = nRT\ [\ln\ V]_{V_1}^{V_2}$$

$$= \text{nRT} [\ln V_2 - \ln V_1] = nRT \ln \frac{V_2}{V_1}$$

$$or\ W_{iso} = 2.303\ \text{nRT}\log\frac{V_2}{V_1} = 2.303\ \text{nRT}\log\frac{P_1}{P_2}$$

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<u>Question:</u> Show that efficiency of a heat engine is $1-\frac{Q_2}{Q_1}$, where Q_1 is heat supplied by source and Q_2 is heat given out into the sink.

Let a working substance in a heat engine absorbs heat Q_1 from a source and rejects heat Q_2 into the sink. So, heat used for performing work is $Q_1 - Q_2$. This must be equal to net work done by the working substance. Hence $W_{net} = Q_1 - Q_2$.

$$\because \eta = \frac{\text{Output work}}{\text{Energy absorbed}}$$

$$\therefore \eta = \frac{W_{net}}{Q_1} = \frac{Q_1 - Q_2}{Q_1} \Rightarrow \boxed{\eta = 1 - \frac{Q_2}{Q_1}}$$

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<u>Question:</u> Show that efficiency of a Carnot engine is given by $\eta = 1-\frac{T_2}{T_1}$, where T_2 and T_1 are temperatures of sink and source respectively.

In a Carnot engine, first step in isothermal expansion. Let the volume of n moles of gas increases from V_1 to V_2 at temperature T_1, then work done by the gas is

$Q_1 = W_1 = 2.303 nRT_1 \log\frac{V_2}{V_1}$, where Q_1 is the heat gained by system

Second step is adiabatic expansion, now let the volume of gas increases from V_2 to V_3 and temperature changes from T_1 to T_2, then work done is

$$W_2 = \frac{1}{1-\gamma}\left(T_2 - T_1\right)$$

Third step is isothermal compression, let the volume of gas changes from V_3 to V_4, then work done is

$Q_2 = W_3 = -2.303nRT_2 \log\frac{V_4}{V_3}$, where Q_2 is the heat loss by the system.

Step 4 is adiabatic compression in which the volume V4 changes back to intial volume V1, then work done is

$$W_4 = -\frac{1}{1-\gamma}\left(T_1 - T_2\right)$$

Net work done is

$$W_{net} = W_{exp} - W_{com} = W_1 + W_2 - \left(W_3 + W_4\right)$$

since $W_2 = W_4$

$$W_{net} = W_1 - W_3 = Q_1 - Q_2$$

Note: You can directly start this derivation from above step also, you can consult your school teacher

Also

For step 1, we can write

$P_1V_1 = P_2V_2$ ....(i)

For step 2

$P_2V_2^{\gamma} = P_3V_3^{\gamma}$ ......(ii)

For step 3

$P_3V_3 = P_4V_4$ ....(iii)

For step 4

$P_4V_4^{\gamma} = P_1V_1^{\gamma}$ ....(iv)

Therefore, we have

$$\cancel{P_1P_2P_3P_4}V_1V_2^{\gamma}V_3V_4^{\gamma} = \cancel{P_1P_2P_3P_4}V_1^{\gamma}V_2V_3^{\gamma}V_4$$

$$(V_1V_3)^{1-\gamma} = (V_2V_4)^{1-\gamma}$$

$$\Rightarrow \frac{V_1}{V_2} = \frac{V_4}{V_3} \quad ....(v)$$

Now,

$$\frac{Q_2}{Q_1} = \frac{-\cancel{2.303nR}T_2\log\left(\frac{V_4}{V_3}\right)}{\cancel{2.303nR}T_1\log\left(\frac{V_2}{V_1}\right)}$$

$$\frac{Q_2}{Q_1} = -\frac{T_2}{T_1}\frac{\log\left(\frac{V_4}{V_3}\right)}{\log\left(\frac{V_2}{V_1}\right)} = \frac{T_2}{T_1}\frac{\log\left(\frac{V_4}{V_3}\right)^{-1}}{\log\left(\frac{V_2}{V_1}\right)} \qquad [\because a\log b = \log b^a]$$

$$\therefore \frac{Q_2}{Q_1} = \frac{T_2}{T_1}\frac{\log\left(\frac{V_3}{V_4}\right)}{\log\left(\frac{V_2}{V_1}\right)}$$

Since $\frac{V_3}{V_4} = \frac{V_1}{V_2}$ [using (v)]

$$\therefore \frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

Since

$$\eta = \frac{W_{net}}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

$$\therefore \boxed{\eta = 1 - \frac{T_2}{T_1}}$$

<u>Question:</u> Show that adiabatic curve is steeper than isothermal curve.

As we know, slope is $\frac{dP}{dV}$

For isothermal process\

$PV = k$

differentiating both sides, we get

$PdV + VdP = 0$

$$\Rightarrow \frac{dP}{dV} = -\frac{P}{V}$$

For adiabatic process, we have

$PV^{\gamma} = k$

differentiating both sides, we get

$P\gamma V^{\gamma-1}dV + V^{\gamma}dP = 0$

$$\Rightarrow \frac{dP}{dV} = -\gamma\frac{P}{V}$$

Clearly slope of adiabatic curve is gamma times more that slope of isothermal curve and since gamma is always greater than 1, so slope of adiabatic curve is more than that of isothermal curve.

Hence an adiabatic curve is always steeper than an isothermal curve.

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