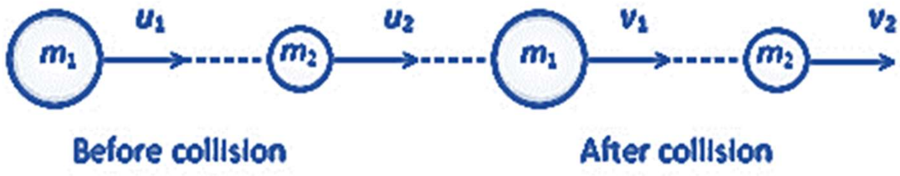
# WORK ENERGY POWER – ALL DERIVATIONS

**Discuss elastic collision in one dimension. Derive an expression for velocities of two bodies after such a collision.**

Consider two bodies of masses $m_1$ and $m_2$ moving with velocities $u_1$ and $u_2$ moving in the same straight line colliding with each other. Let their velocities be $v_1$ and $v_2$ after the collision.




Since momentum remains conserved in an elastic collision, therefore

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$\Rightarrow m_1u_1 - m_1v_1 = +m_2v_2 - m_2u_2$$
$$\Rightarrow m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad .......(i)$$

As kinetic energy is also conserved in elastic collision therefore

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$
$$\Rightarrow \frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_2u_2^2$$
$$\Rightarrow \frac{1}{2}m_1(u_1^2 - v_1^2) = \frac{1}{2}m_2(v_2^2 - u_2^2)$$
$$\Rightarrow \cancel{\frac{1}{2}}m_1(u_1 - v_1)(u_1 + v_1) = \cancel{\frac{1}{2}}m_2(v_2 - u_2)(v_2 + u_2) .......(ii)$$

From (i) and (ii), we get

$$\frac{\cancel{m_1}\,\cancel{(u_1 - v_1)}(u_1 + v_1)}{\cancel{m_1}\,\cancel{(u_1 - v_1)}} = \frac{\cancel{m_2}\,\cancel{(v_2 - u_2)}(v_2 + u_2)}{\cancel{m_2}\,\cancel{(v_2 - u_2)}}$$
$$\Rightarrow u_1 + v_1 = v_2 + u_2$$
$$\Rightarrow u_1 - u_2 = v_2 - v_1 \quad .............(iii)$$

Thus, relative velocity of approach = relative velocity of separation

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Since

$$e(\text{coefficient of restitution}) = \frac{\text{relative velocity of seperation}}{\text{relative velocity of approach}}$$

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

Therefore, for perfectly elastic collision, e = 1

Now, from (iii), we get

$v_2 = u_1 - u_2 + v_1$, putting this in momentum conservation equation, we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2(u_1 - u_2 + v_1)$$
$$\Rightarrow m_1u_1 + m_2u_2 = m_1v_1 + m_2u_1 - m_2u_2 + m_2v_1$$
$$\Rightarrow (m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$
$$\Rightarrow \boxed{v_1 = \frac{(m_1 - m_2)u_1}{(m_1 + m_2)} + \frac{2m_2u_2}{(m_1 + m_2)}}$$

Similarly, we can prove that

$$\boxed{v_2 = \frac{(m_2 - m_1)u_2}{(m_1 + m_2)} + \frac{2m_1u_1}{(m_1 + m_2)}}$$

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**Derive an expression for the elastic potential energy of a stretched spring.**

Consider a spring of spring constant k. Let one end of this spring is fixed and a force F is applied on the other end to stretch its length by small amount dx. Then, work done is

$$dW = \vec{F}.\overrightarrow{dx}$$
$$\Rightarrow dW = -(kx)dx\cos 180^\circ$$
$$\Rightarrow dW = -kx(-1)dx$$
$$\Rightarrow dW = kxdx$$

Total work done in stretching the spring from 0 to $x_o$

$$W = \int_0^{x_o} kxdx$$

$$\Rightarrow W = k\int_0^{x_o} xdx$$

$$\Rightarrow W = k\left[\frac{x^2}{2}\right]_0^{x_o}$$

$$\Rightarrow W = \frac{k}{2}\left[x_o^2 - (0)^2\right]$$

$$\Rightarrow W = \frac{1}{2}kx_o^2$$

This work is stored in the spring in the form of elastic potential energy, so

$$\boxed{U = \frac{1}{2}kx_o^2}$$

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## State and prove the work energy theorem.

Consider a body of mass m moving with a velocity $v_i$. Now let a force F is applied to it and its velocity becomes $v_f$ after some time. Let the velocity change be dv for small time dt and body travels a distance ds during this time, then work done dW is

$$dW = Fds$$

$$\Rightarrow dW = (ma)ds$$

$$\Rightarrow dW = m\frac{dv}{dt}ds$$

$$\Rightarrow dW = mdv\left(\frac{ds}{dt}\right)$$

$$\Rightarrow dW = mvdv$$

So, total work done when velocity changes from $v_i$ to $v_f$

$$\Rightarrow \int dW = \int_{v_i}^{v_f} mvdv$$

$$\Rightarrow W = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f}$$

$$\Rightarrow W = m\left[\frac{v_f^2}{2} - \frac{v_i^2}{2}\right]$$

$$\Rightarrow W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

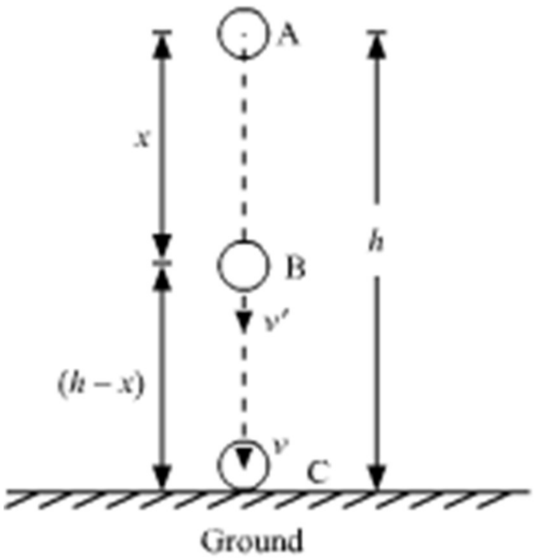
**Prove that a body falling freely the total mechanical energy always remains conserved.**

Consider a body of mass m falling form a height h. Consider three points A, B and C in its path as shown. Now

Total energy of body at A is only potential as its velocity is zero. Therefore

$$T.E_A = mgh$$

At B, body has both potential and kinetic energy. Since the body has covered a distance x, therefore its velocity at B is




$$v_B^2 - (0)^2 = 2gx$$
$$\Rightarrow v_B^2 = 2gx$$

So total energy of body at B is

$$TE_B = mg(h-x) + \frac{1}{2}mv_B^2$$
$$\Rightarrow TE_B = mg(h-x) + \frac{1}{2}m(2gx)$$
$$\Rightarrow TE_B = mgh - mgx + mgx$$
$$\Rightarrow TE_B = mgh$$

At point C, body has only kinetic energy. Velocity of body at C is

$$v_C^2 - (0)^2 = 2gh$$
$$\Rightarrow v_C^2 = 2gh$$

So, total energy of body at C is

$$TE_C = \frac{1}{2}mv_C^2$$
$$\Rightarrow TE_C = \frac{1}{2}m(2gh)$$
$$\Rightarrow TE_C = mgh$$

Clearly, $TE_A = TE_B = TE_C$

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**Derive a formula for kinetic energy of a body of mass m moving with velocity v.**

Consider a body of mass m kept at rest. Now let a force F is applied to it and its velocity becomes v after some time. Let the velocity change be dv for small time dt and body travels a distance ds during this time, then work done dW is

$$dW = Fds$$
$$\Rightarrow dW = (ma)ds$$
$$\Rightarrow dW = m\frac{dv}{dt}ds$$
$$\Rightarrow dW = mdv\left(\frac{ds}{dt}\right)$$
$$\Rightarrow dW = mvdv$$

So, total work done in change velocity from 0 to v is

$$W = \int_0^v mvdv$$
$$\Rightarrow W = m\left[\frac{v^2}{2}\right]_0^v$$
$$\Rightarrow W = \frac{1}{2}m\left(v^2 - (0)^2\right)$$
$$\Rightarrow W = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2$$
$$\Rightarrow \boxed{W = \frac{1}{2}mv^2}$$

This work is stored in the body in the form of Kinetic energy. This $KE = \frac{1}{2}mv^2$

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**Derive an expression for common velocity of two bodies after perfectly inelastic collision.**

Consider two bodies of masses $m_1$ and $m_2$ moving with velocities $v_1$ and $v_2$ collide inelastically and stick together with each other. Let their common velocity after collision be v. As momentum remains conserved in inelastic collision, therefore,

$$m_1v_1 + m_2v_2 = (m_1v + m_2v)$$
$$v(m_1 + m_2) = m_1v_1 + m_2v_2$$
$$\boxed{v = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}}$$

**Derive an expression for loss of energy when two bodied of masses m1 and m2 moving with velocities v1 and v2 undergo inelastic collision.**

**Total KE of the bodies before collision**

$$KE_i = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Total KE after collision:

$$KE_f = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 \text{ (as velocity becomes equal after inelastic collision)}$$

Therefore, loss of kinetic energy is

$$KE_i - KE_f$$

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1v_1 + m_2v_2}{m_1 + m_2}\right)^2$$

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}\cancel{(m_1 + m_2)}\frac{(m_1v_1 + m_2v_2)^2}{(m_1 + m_2)^{\cancel{2}}}$$

$$= \frac{1}{2}\left[\frac{m_1v_1^2(m_1 + m_2) + m_2v_2^2(m_1 + m_2) - (m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2)}{(m_1 + m_2)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{m_1^2v_1^2} + m_1m_2v_1^2 + m_1m_2v_2^2 + \cancel{m_2^2v_2^2} - \cancel{m_1^2v_1^2} - \cancel{m_2^2v_2^2} - 2m_1m_2v_1v_2}{(m_1 + m_2)}\right]$$

$$= \frac{1}{2}m_1m_2\left[\frac{(v_1^2 + v_2^2 - 2v_1v_2)}{(m_1 + m_2)}\right]$$

$$\boxed{\Delta KE = \frac{1}{2}m_1m_2\frac{(v_1 - v_2)^2}{(m_1 + m_2)}}$$

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