Waves – all derivations

Speed of transverse wave on stretched string

Speed v of a transverse wave depends upon tension in the string and linear mass density (mass/length)

$$v \propto T^a \lambda^b$$
$$\Rightarrow v = kT^a \lambda^b$$
$$\Rightarrow [LT^{-1}] = k[MLT^{-2}]^a [ML^{-1}]^b$$
$$\Rightarrow [LT^{-1}] = k[M^{a+b}L^{a-b}T^{-2a}]$$
$$a + b = 0, a - b = 1,$$
$$\therefore a = \frac{1}{2}, b = -\frac{1}{2}$$

$$\boxed{\therefore v = k\sqrt{\frac{T}{\lambda}}}$$

****

Speed of longitudinal wave in a gas

Speed v of a transverse wave depends upon bulk modulus B of the gas and density of the gas ρ

$$v \propto B^a \rho^b$$
$$\Rightarrow v = kB^a \rho^b$$
$$\Rightarrow [LT^{-1}] = k[ML^{-1}T^{-2}]^a [ML^{-3}]^b$$
$$\Rightarrow [LT^{-1}] = k[M^{a+b}L^{-a-3b}T^{-2a}]$$
$$a + b = 0, -a - 3b = 1,$$
$$\therefore a = \frac{1}{2}, b = -\frac{1}{2}$$

$$\boxed{\therefore v = k\sqrt{\frac{B}{\rho}}}$$

****

Equation of plane progressive wave

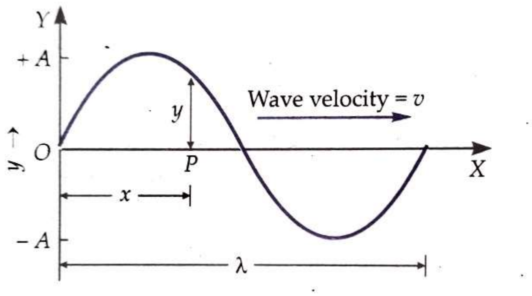
Suppose a simple harmonic wave starts from the origin O and travels along the positive direction of X-axis with speed v. Let the time be measured from the instant when the particle at the origin O is passing through the mean position. Taking the initial phase of the particle to be zero, the displacement of the particle at the origin O (x = 0) at any instant t is given by

$y(0,t) = A \sin \omega t$ .... (i)

Where T is the periodic time and A is the amplitude of the wave.

Consider a particle P on x axis at a distance x from O.

The disturbance starting from the origin O will reach P in $\frac{x}{v}$ seconds later than the particle at O. Therefore



Displacement of the particle at P at any instant t = Displacement of the particle at O at a $\frac{x}{v}$ seconds earlier

= Displacement of the particle at O at time $\left(t - \frac{x}{v}\right)$

Thus the displacement of the particle at P at any time t can be obtained by replacing t by $\left(t - \frac{x}{v}\right)$ in equation (i)

$$y(x,t) = A \sin \omega\left(t - \frac{x}{v}\right) = A \sin\left(\omega t - \frac{\omega}{v}x\right)$$

But $\frac{\omega}{v} = \frac{2\pi\nu}{v} = \frac{2\pi}{\lambda} = k$

The quantity $k = \frac{2\pi}{\lambda}$ is called angular wave number. Hence,

$$\boxed{y(x,t) = A \sin(\omega t - kx)}$$

****

<u>Relation between particle velocity and wave velocity</u>

particle velocity $v_p = \frac{dy}{dt}$; wave veocity $v_w = \frac{dx}{dt}$

$$v_p = A\omega\cos(\omega t - kx)$$

$$\frac{dy}{dx} = -A\cos(\omega t - kx)k$$

$$\therefore \frac{v_p}{\left(\frac{dy}{dx}\right)} = -\frac{A\omega\cos(\omega t - kx)}{A\cos(\omega t - kx)k} = -\frac{\omega}{k}$$

$$v_p = -\frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} \times \frac{dy}{dx} = -\frac{\lambda}{T} \times \frac{dy}{dx}$$

$$\Rightarrow \boxed{v_p = -v_w \times \frac{dy}{dx}}$$

****

Newton's formula for velocity of sound in gases and Laplace's correction

Newton assumed that the sound waves travel through air under isothermal conditions. He argued that the small amount of heat produced in a compression is rapidly conducted to the surrounding rarefaction where slight cooling is produced. Thus the temperature of the gas remains constant.

For isothermal change

PV = constant

Differentiating both side, we get

$$PdV + VdP = 0$$

$$\Rightarrow PdV = -VdP$$

$$\Rightarrow P = -\frac{dPV}{dV} = B$$

Where B is the bulk modulus of the gas.

Now, since velocity v of a longitudinal wave in medium is given by $v = \sqrt{\frac{B}{\rho}}$, where ρ is the density of the medium, therefore

$$v = \sqrt{\frac{P}{\rho}} = v = \sqrt{\frac{101325}{1.293}} = 280\,ms^{-1}$$

Which is incorrect having an error of 16%

Laplace pointed out that the sound travels through a gas under adiabatic conditions not under isothermal conditions because

- Compression and rarefactions are so rapid that there is no time for exchange of heat.
- Air is an insulator so free exchange of heat is not possible.

So, applying the equation of state for an adiabatic process, we get

$PV^{\gamma} = K$

$\Rightarrow P\gamma V^{\gamma-1}dV + V^{\gamma}dP = 0$

$\Rightarrow \gamma P\dfrac{V^{\gamma}}{V}dV + V^{\gamma}dP = 0$

$\Rightarrow V^{\gamma}\left[\dfrac{\gamma P}{V}dV + dP\right] = 0$

$\Rightarrow \gamma P = -\dfrac{dPV}{dV} = B$

$\therefore v = \sqrt{\dfrac{\gamma P}{\rho}} = \sqrt{1.4} \times 280\,ms^{-1} = 331.3ms^{-1}$, which the correct value of velocity of sound in air.

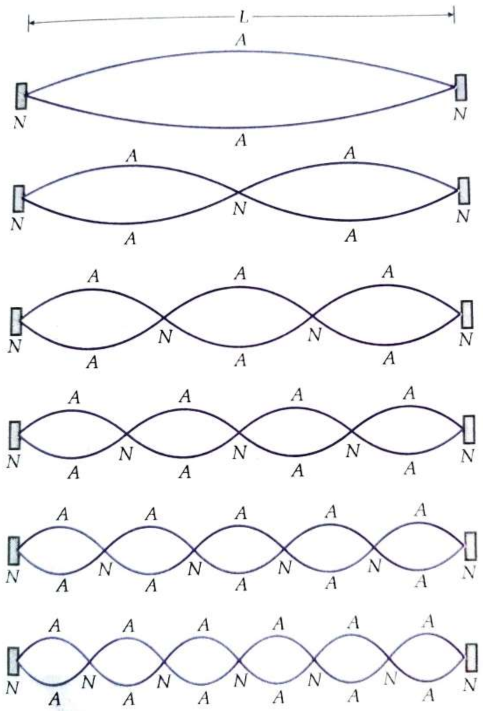
****

Standing waves on stretched strings

Q: Discuss the formation of standing waves in a string fixed at both ends and the different modes of vibrations

Or

Discuss the formation of harmonics in a stretched string. Show that in case of a stretched string in the four harmonics are in the ratio 1 : 2: 3 : 4.



Standing waves on stretched strings

Consider a wave travelling along the string given by

$y_1 = A\sin(\omega t - kx)$

After reflection from the rigid end the equation of the reflected wave is given by

$y_2 = A\sin(\omega t + kx + \pi)$

or

$y_2 = -A\sin(\omega t + kx)$

When these two waves superimpose, then the resultant wave is given by

$y_1 + y_2 = A\sin(\omega t - kx) - A\sin(\omega t + kx)$

$y = A\left\{2\sin\left(\dfrac{\omega t - kx - \omega t - kx}{2}\right)\cos\left(\dfrac{\omega t - kx + \omega t + kx}{2}\right)\right\}$

$y = 2A\sin\left(\dfrac{-2kx}{2}\right)\cos\left(\dfrac{2\omega t}{2}\right)$

$y = -2A\sin kx\cos\omega t$

As there is always a node at the end, so if length of the rope is L then we can say when x = L, y = 0

$$0 = 2A\sin kL \sin\omega t$$
$$\sin kL = \sin n\pi$$
$$kL = n\pi$$
$$\frac{2\pi}{\lambda}L = n\pi$$
$$L = \frac{n\lambda}{2}$$

For each value of n, there is a corresponding value of λ, so we can write $\frac{2\pi L}{\lambda_n} = n\pi$ or $\lambda = \frac{2L}{n}$

The speed of transverse wave on a string of linear mass density m is given by $v = \sqrt{\frac{T}{m}}$

Kindly note here m is mass per unit length of the rope, not mass

So the frequency of vibration of the strings is

$$\nu_n = \frac{v}{\lambda_n} = \frac{n}{2L}\sqrt{\frac{T}{m}}$$

For n = 1, $v_1 = \frac{1}{2L}\sqrt{\frac{T}{m}} = \nu$ (say)

This is the lowest frequency with which the string can vibrate and is called fundamental frequency or first harmonic.

$$\text{For } n = 2,\quad \nu_2 = \frac{2}{2L}\sqrt{\frac{T}{m}} = 2\nu \quad \text{(first ovetone or second harmonic)}$$

$$\text{For } n = 3,\quad \nu_3 = \frac{3}{2L}\sqrt{\frac{T}{m}} = 3\nu \quad \text{(second ovetone or third harmonic)}$$

$$\text{For } n = 2,\quad \nu_4 = \frac{4}{2L}\sqrt{\frac{T}{m}} = 4\nu \quad \text{(third ovetone or fourth harmonic)}$$

Position of nodes

$$x = 0, \frac{L}{n}, \frac{2L}{n}, \ldots\ldots\ldots\ldots, L$$

Position of antinodes

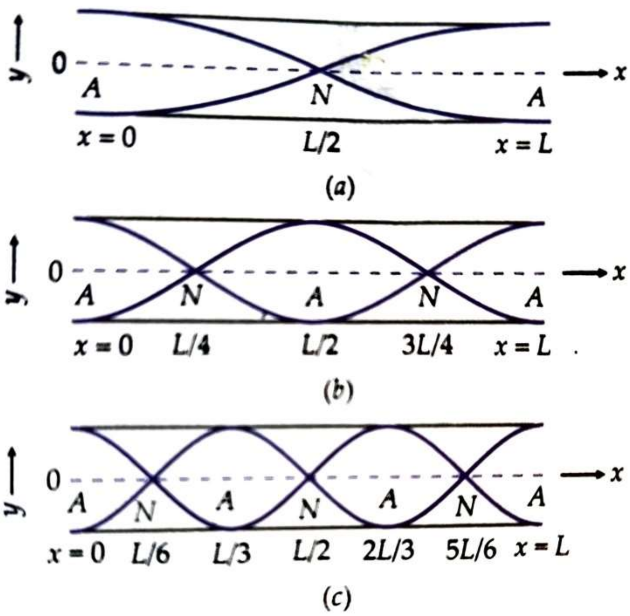
$$x = \frac{L}{2n}, \frac{3L}{2n}, \frac{5L}{2n}, \ldots\ldots\ldots, \frac{(2n-1)L}{2n}$$

****

<u>Standing waves in organ pipes</u>

First mode of vibration

In the simplest mode of vibration, there is one node in the middle and to antinodes at the ends of the pipe.



Here length of the pipe,

$$L = 2.\frac{\lambda_1}{4} = \frac{\lambda_1}{2}$$

$$\therefore \lambda_1 = 2L$$

Frequency of vibration,

$$\nu_1 = \frac{v}{\lambda_1} = \frac{1}{2L}\sqrt{\frac{\gamma P}{\rho}} = \nu$$

This is called fundamental frequency or first harmonic.

Second mode of vibration

Here antinodes at the open ends are separated by two nodes and one antinode.

$$\lambda = 4\frac{\lambda_2}{4} = \lambda_2$$

Frequency, $\nu_2 = \frac{v}{\lambda_2} = \frac{1}{L}\sqrt{\frac{\gamma P}{\rho}} = 2\nu$

This frequency is called first overtone or second harmonic.

Third mode of vibration

Here the antinodes at the open ends are separated by three nodes and two antinodes.

$$L = 6\frac{\lambda_3}{4} \text{ or } \lambda_3 = \frac{2L}{3}$$

$\therefore$ Frequency, $\nu_3 = \frac{v}{\lambda_3} = \frac{3}{2L}\sqrt{\frac{\gamma P}{\rho}} = 3\nu$

This frequency is called the second harmonic or third harmonic

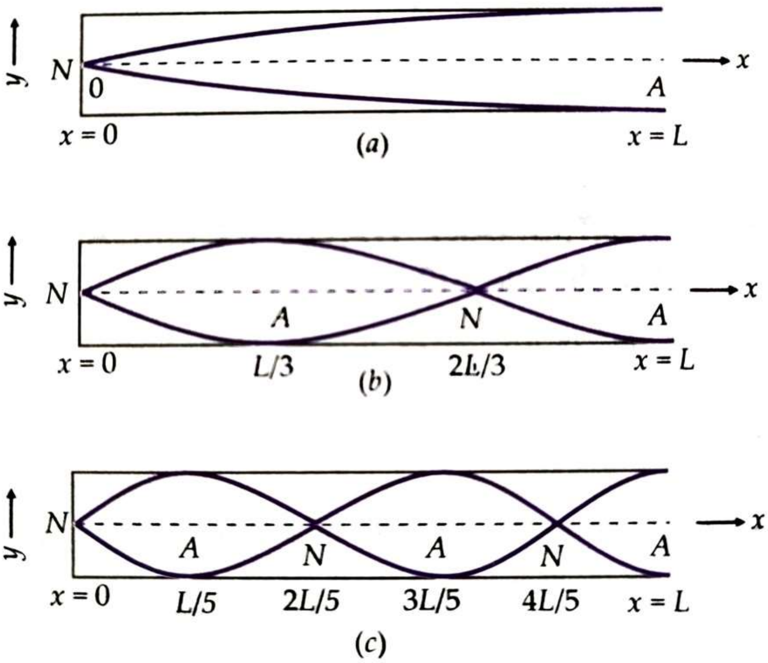
Similarly $\nu_n = \frac{v}{\lambda_{3n}} = \frac{n}{2L}\sqrt{\frac{\gamma P}{\rho}} = n\nu$

Hence the various frequencies of an open organ pipe are in the ratio 1:2:3:4….these are called harmonics.

Closed organ pipes

First mode of vibration

In the simplest mode of vibration, there is only one node at the closed end and one antinode at the open end. If L is the length of the organ pipe, then



$$L = \frac{\lambda_1}{4} \text{ or } \lambda_1 = 4L$$

Frequency,

$$\nu_1 = \frac{v}{\lambda_1} = \frac{1}{4L}\sqrt{\frac{\gamma P}{\rho}} = \nu$$

This is called first harmonic or fundamental frequency.

Second mode of vibration

In this mode of vibration, there is one node and one antinode between a node at the closed end and an antinode at the open end

$$L = \frac{3\lambda_2}{4} \text{ or } \lambda_2 = \frac{4L}{3}$$

Frequency,

$$\nu_2 = \frac{v}{\lambda_2} = \frac{3}{4L}\sqrt{\frac{\gamma P}{\rho}} = 3\nu$$

This frequency is called first overtone or third harmonic.

Third mode of vibration

In this mode of vibration, there are two nodes and two antinodes between a node at the closed end and an antinode at the open end.

$$L = \frac{5\lambda_3}{4} \text{ or } \lambda_3 = \frac{4L}{5}$$

Frequency,

$$\nu_3 = \frac{v}{\lambda_3} = \frac{5}{4L}\sqrt{\frac{\gamma P}{\rho}} = 5\nu$$

Hence different frequencies produced in a closed organ pipe are in the ratio 1 : 3 : 5 : 7 …..i.e. only odd harmonics are present in a closed organ pipe.

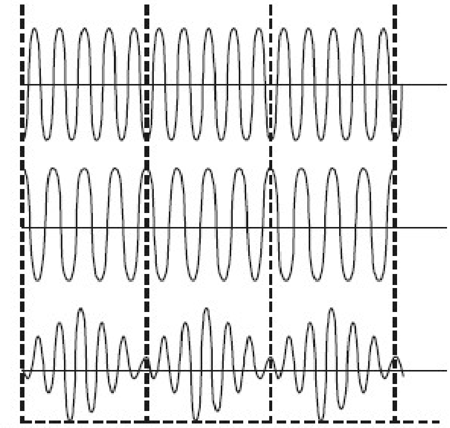
****

<u>Beats: expression for beat interval and beat frequency</u>

The periodic variations in the intensity of sound caused by superposition of two sound waves of slightly different frequencies are called beats.

Consider two harmonic waves of frequencies ν_1 and ν_2 (let $\nu_1 > \nu_2$) and each of amplitude A travelling in a medium in the same direction. The displacements of due to two waves are given as

$$y_1 = A\sin\omega_1 t = A\sin 2\pi\nu_1 t \text{ and}$$
$$y_2 = A\sin\omega_2 t = A\sin 2\pi\nu_2 t$$



By the principle of superposition, the resultant displacement at a given point will be

$$y = y_1 + y_2 = A\sin 2\pi\nu_1 t + A\sin 2\pi\nu_2 t$$
$$= 2A\cos 2\pi\left(\frac{\nu_1 - \nu_2}{2}\right)t\sin 2\pi\left(\frac{\nu_1 + \nu_2}{2}\right)t$$

If we write

$\nu_a = \dfrac{\nu_1 - \nu_2}{2}$ and $\nu_b = \dfrac{\nu_1 + \nu_2}{2}$, then

$$y = 2A\cos 2\pi\nu_a t\sin 2\pi\nu_b t$$

Amplitude of this wave is $2A\cos 2\pi\nu_a t$, this amplitude is maximum when

$$\cos 2\pi\nu_a t = \pm 1$$
$$\cos 2\pi\nu_a t = \cos n\pi$$
$$\Rightarrow 2\pi\nu_a t = n\pi$$
$$\Rightarrow t = \frac{n}{\nu_1 - \nu_2}$$

This is maximum for $t_1 = \dfrac{1}{\nu_1 - \nu_2}$, $t_2 = \dfrac{2}{\nu_1 - \nu_2}$.....

Therefore time interval between two maximum $t_2 - t_1 = \dfrac{1}{\nu_1 - \nu_2}$

And $2A\cos 2\pi\nu_a t$ is minimum when

$$\cos 2\pi\nu_a t = 0$$

$$\Rightarrow \cos 2\pi\nu_a t = \cos(2n+1)\frac{\pi}{2}$$

$$\Rightarrow 2\pi\nu_a t = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow t = \frac{(2n+1)}{2(\nu_1 - \nu_2)}$$

This is minimum for $t_1 = \frac{1}{2(\nu_1 - \nu_2)},\ t_2 = \frac{3}{2(\nu_1 - \nu_2)}......$

Therefore time interval between successive minima is $t_2 - t_1 = \frac{3}{2(\nu_1 - \nu_2)} - \frac{1}{2(\nu_1 - \nu_2)} = \frac{1}{(\nu_1 - \nu_2)}$

Since one maxima and one minima make one beat, therefore

Beat interval is $\frac{1}{(\nu_1 - \nu_2)}$

Beat frequency = $(\nu_1 - \nu_2)$

****

A small donation will help and motivate us to make more quality content for free for all the needy students of our country.

Your small amount can make a big difference in someone's life ☺