Gravitation

Most Important Questions Answers

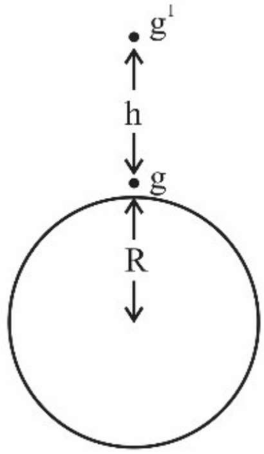
Question (1): What is acceleration due to gravity? Derive a formula for it in terms of mass and radius of a given planet.

Consider a small body of mass m kept near the surface of earth. The weight of this body is given by

W = mg

Also, the gravitational force of earth on the body is

$F = \frac{GMm}{R^2}$, where M is the mass of the earth and R is the radius of the earth.



(As the size of body is very small as compared to size of earth, we have considered the distance between the centre of body and earth as radius of earth).

Therefore, we can write

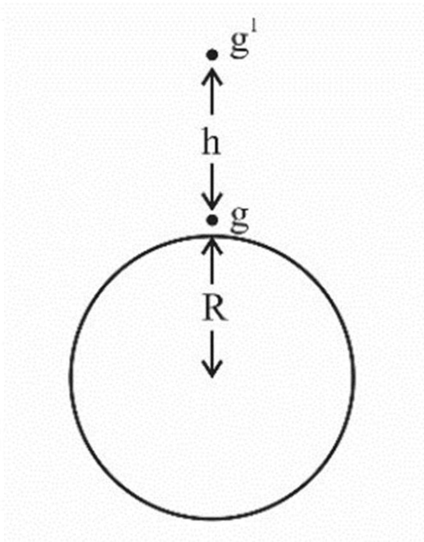
$$mg = \frac{GMm}{R^2}$$

$$\Rightarrow \boxed{g = \frac{GM}{R^2}}$$ which is the formula for acceleration due to gravitation.

As $M = 6\times10^{24}\,kg, R = 6.4\times10^{6}\,m$, putting all these values, we get g = 9.8 ms^{-2}.

Question (2): Discuss the variation of acceleration due to gravity with height. Hence, derive a formula for percentage change in acceleration due to gravity for small heights.

As acceleration due to gravity at surface of earth, $g = \frac{GM}{R^2}$. Therefore, acceleration due to gravity at a height h above the surface of earth $g' = \frac{GM}{(R+h)^2}$.



$$\therefore \frac{g'}{g} = \frac{\frac{GM}{(R+h)^2}}{\frac{GM}{R^2}}$$

$$\Rightarrow \frac{g'}{g} = \frac{R^2}{(R+h)^2} \qquad \text{. . . . . (i)}$$

$$\Rightarrow \frac{g'}{g} = \frac{R^2}{R^2(R+\frac{h}{R})^2}$$

$$\Rightarrow \frac{g'}{g} = \frac{1}{(R + \frac{h}{R})^2}$$

$$\Rightarrow \frac{g'}{g} = \left(1 + \frac{h}{R}\right)^{-2}$$

If h << R, we can expand above expression using binomial theorem

$$\frac{g'}{g} = \left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow 1 - \frac{g'}{g} = \frac{2h}{R}$$

$$\Rightarrow \frac{g - g'}{g} \times 100 = \frac{2h}{R} \times 100$$

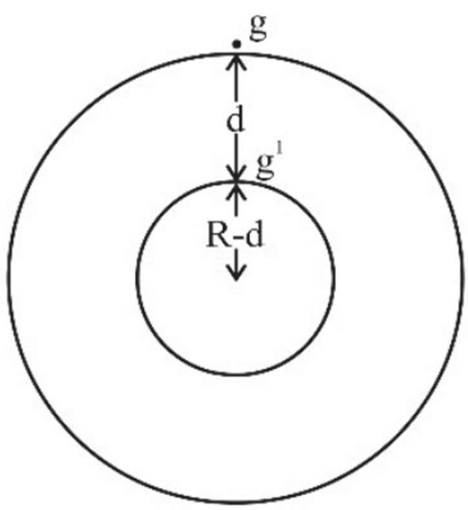
i.e. percentage decrease in the value of g at a height h = $\frac{2h}{R} \times 100$. Where h is very small as compared to radius of earth.

Question (3): Discuss the variation of acceleration due to gravity with depth. Hence, derive a formula for percentage change in acceleration due to gravity for a depth d below the surface of earth.

If body is taken at a depth d below the surface of earth then, $g' = \frac{GM'}{(R-d)^2}$.

Where M' is the mass of that spherical part of earth whose radius is $(R-d)$.

Let earth be a uniform sphere of density ρ, then



$$M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore g = \frac{G}{R^2}\frac{4}{3}\pi R^3 \rho$$

$$\Rightarrow g = \frac{4}{3}G\pi R\rho$$

And,

$$M' = \frac{4}{3}\pi(R-d)^3 \rho$$

$$\therefore g' = \frac{G}{(R-d)^2}\frac{4}{3}\pi(R-d)^3 \rho$$

$$\Rightarrow g' = \frac{4}{3}G\pi(R-d)\rho$$

Therefore,

$$\frac{g'}{g}=\frac{\frac{4}{3}G\pi(R-d)\rho}{\frac{4}{3}G\pi R\rho}$$

$$\Rightarrow \frac{g'}{g}=\frac{R-d}{R} \Rightarrow \frac{g'}{g}=1-\frac{d}{R}$$

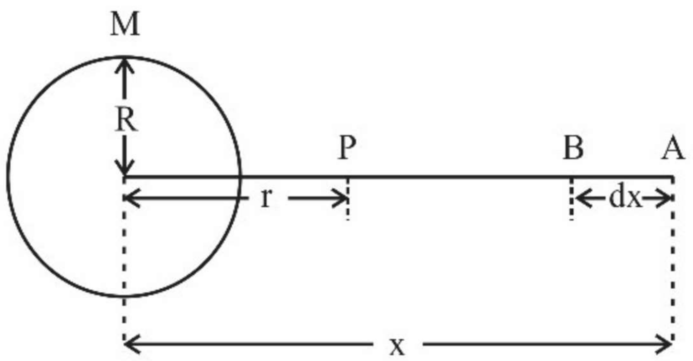
Hence, percentage decrease at a depth d in the value of g is given by $\left(1-\frac{d}{R}\right)\times 100$.

Question (4): Define gravitational potential at a point. Derive an expression for gravitational potential at a distance r due to a mass M.

What do you mean by gravitational potential energy of a system? Derive an expression for gravitational potential energy of two bodies of masses M and m with their centres separated by a distance r.

Consider a body of mass M. A body of mass m is place at a distance *x* from the centre of this body.

Then force of gravitation between them is, $F=\frac{GMm}{x^2}$.



Small amount of work (dW) done to move mass m through small distance d*x* towards mass M is

$$dW = F dx$$

$$\Rightarrow dW=\frac{GMm}{x^2}dx$$

Therefore, total work done to move this body from x = ∞ to *x* = r is given by

$$\int dW=\int_{\infty}^{r}\frac{GMm}{x^2}dx$$

$$\Rightarrow W=GMm\int_{\infty}^{r}\frac{1}{x^2}dx$$

$$\Rightarrow W=GMm\left[\frac{x^{-2+1}}{-2+1}\right]_{\infty}^{r}$$

$$\Rightarrow W=-GMm\left[\frac{1}{x}\right]_{\infty}^{r}$$

$$\Rightarrow W = -GMm\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$\Rightarrow W = -\frac{GMm}{r}$$

By definition, potential at P is $V = \frac{W}{m}$. Therefore,

$$\boxed{V = -\frac{GM}{r}}$$

Clearly, the work done to arrange this system is $W = -\frac{GMm}{r}$. This work is stored in the system in the form of potential energy.

In general, potential energy of a system of two bodies of masses m_1 and m_2 placed at a distance r apart is given by

$$\boxed{U = -\frac{Gm_1m_2}{r}}.$$

Question (5): Prove that the work done to move a body of mass m through a distance h against the gravitational force is mgh

For a body of mass m placed at the surface of earth, the potential energy of the system is given by

$$U_1 = -\frac{GMm}{R}$$

Now, if this body is taken to a height h, potential energy is given by

$$U_2 = -\frac{GMm}{R+h}$$

Change in potential energy is $\Delta U = U_2 - U_1$

$$\Rightarrow \Delta U = -\frac{GMm}{R+h} + \frac{GMm}{R}$$

$$\Rightarrow \Delta U = \frac{GMm}{R}\left(1 - \frac{1}{1+\frac{h}{R}}\right)$$

$$\Rightarrow \Delta U = \frac{GMm}{R}\left(1 - \left(1+\frac{h}{R}\right)^{-1}\right)$$

For small heights, above expression can expanded binomially and it can be written as

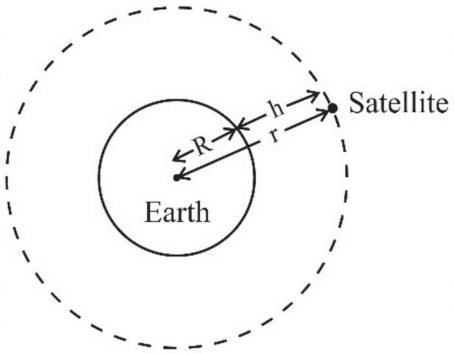
$$\Rightarrow \Delta U = \frac{GMm}{R}\left(1 - \left(1-\frac{h}{R}\right)\right)$$

$$\Rightarrow \Delta U = \frac{GMm}{R}\left(\frac{h}{R}\right)$$

$$\Rightarrow \Delta U = \frac{GMmh}{R^2} \Rightarrow \boxed{\Delta U = mgh}$$

Question (6): What is a satellite. What is the principle for launching a satellite? Derive an expression for the orbital velocity of stallelite.

As, centripetal force = Gravitational force



$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{GM}{r}}}$$

$$\Rightarrow v = \sqrt{\frac{GM}{R+h}}$$

If satellite is revolving very close to earth, then we can write above expression as

$$v = \sqrt{\frac{GM}{R}}$$

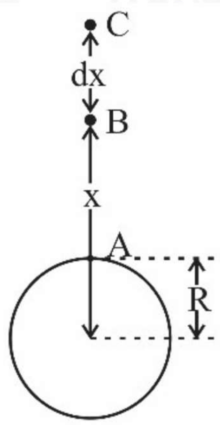
Putting all the values in the above expression, we get

v = 7.92 kms^{-1}.

Question (7): What do you mean by escape velocity? Derive an expression for escape velocity in terms of parameters of a given planet.

Consider a body of mass m at a distance *x* from the centre of earth as shown. Force acting on this body is

$$F = \frac{GMm}{x^2}.$$



Small amount of work (dW) done to move mass m through small distance dx away from earth is

$$dW = Fdx$$

$$\Rightarrow dW = \frac{GMm}{x^2}dx$$

Therefore, total work done to move this body from x = R to *x* = ∞ is given by

$$\Rightarrow W = GMm\left[\frac{x^{-2+1}}{-2+1}\right]_R^{\infty} \Rightarrow W = -GMm\left[\frac{1}{x}\right]_R^{\infty}$$

$$\Rightarrow W = -GMm\left[\frac{1}{\infty} - \frac{1}{R}\right]$$

$$\Rightarrow W = \frac{GMm}{R}$$

This work must be equal to KE given to the body at the time of launch

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

$$\Rightarrow \boxed{v_e = \sqrt{\frac{2GM}{R}}}$$

Alternate method

Since total energy of the body at the surface and at infinity must be equal. Therefore, we can write

$$\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R}\right) = 0 + 0$$

$$\Rightarrow \boxed{v_e = \sqrt{\frac{GMm}{R}}}$$

Question (8): State three laws of Kepler's planetary motion.

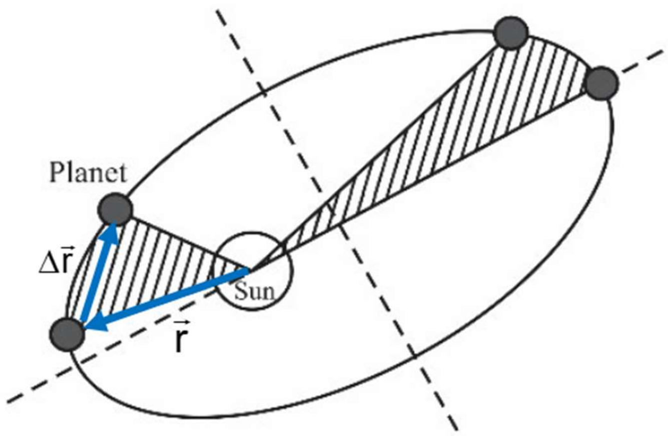
First law (law of orbits)

Orbits of planets are elliptical in shape and sun is situated at one of the foci.

Second law (Law of areas)

The area swept by planet per unit time with respect to sun i.e. areal velocity of a planet around sun is always constant.

Proof:



$$\Delta A = \frac{1}{2}\left(\vec{r} \times \overrightarrow{\Delta r}\right)$$

$$\because \vec{v} = \frac{\overrightarrow{\Delta r}}{\Delta t} \Rightarrow \overrightarrow{\Delta r} = \vec{v}\Delta t$$

$$\therefore \Delta A = \frac{1}{2}\left(\vec{r} \times \vec{v}\Delta t\right)$$

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\right)$$

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2m}\left(\vec{r} \times \vec{p}\right)$$

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2m} \times \vec{L}$$

since $\vec{L} = \text{constant}$

$$\therefore \frac{\Delta A}{\Delta t} = \text{constant}$$

Third law (law of time periods):

The square of time period of a planet around sun is directly proportional to the cube of its average orbital radius.

Proof:

Let mass of planet be m and mass of sun be M and average orbital radius of planet is r, then

Centripetal force = gravitational force

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow v^2 = \frac{GM}{r}$$

$$\Rightarrow \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

$$\Rightarrow \frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$\Rightarrow T^2 = \frac{4\pi^2}{GM} r^3$$

Since $\frac{4\pi^2}{GM}$ is constant, therefore $T^2 \propto r^3$

For two planets having time periods T_1 and T_2

and orbital radii R_1 and R_2, we can write

$$\boxed{\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}}$$