Motion in a Straight Line

1 Equations of Motion using Calculus Method

In kinematics, the three equations of motion describe the relationship between velocity, acceleration, time, and displacement for an object moving under uniform acceleration. These equations can be derived using **differential and integral calculus**.

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Equation of motion using calculus method

1.1 First Equation of Motion: $v = u + at$

Acceleration is defined as the rate of change of velocity:

$$a = \frac{dv}{dt}$$

Rearranging:

$$dv = a\,dt$$

Integrating both sides from initial velocity (u) to final velocity (v) over the time interval 0 to t:

$$\int_u^v dv = \int_0^t a\,dt$$

Solving the integrals:

$$[v]_u^v = a[t]_0^t$$

$$v - u = at$$

$$v = u + at$$

Explanation: The first equation of motion states that the final velocity (v) is equal to the initial velocity (u) plus the product of acceleration (a) and time (t).

1.2 Second Equation of Motion: $s = ut + \frac{1}{2}at^2$

Velocity is the rate of change of displacement:

$$v = \frac{ds}{dt}$$

Substituting $v = u + at$:

$$\frac{ds}{dt} = u + at$$

Rearranging:

$$ds = (u + at)\,dt$$

Integrating both sides from displacement 0 to s over the time interval 0 to t:

$$\int_0^s ds = \int_0^t (u + at)\,dt$$

Solving the integrals:

$$[s]_0^s = \left[ut + \frac{1}{2}at^2\right]_0^t$$

$$s = ut + \frac{1}{2}at^2$$

Explanation: The second equation of motion expresses the displacement (s) in terms of initial velocity (u), acceleration (a), and time (t).

1.3 Third Equation of Motion: $v^2 = u^2 + 2as$

We use the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$$

Rearranging:

$$a\,ds = v\,dv$$

Integrating both sides from $s = 0$ to $s = s$ and $v = u$ to $v = v$:

$$\int_u^v v\,dv = \int_0^s a\,ds$$

Solving the integrals:

$$\left[\frac{v^2}{2}\right]_u^v = a[s]_0^s$$

$$\frac{v^2}{2} - \frac{u^2}{2} = as$$

Multiplying both sides by 2:

$$v^2 = u^2 + 2as$$

Explanation: The third equation of motion relates the final velocity (v) to initial velocity (u), acceleration (a), and displacement (s) without involving time.

These derivations confirm the three fundamental kinematic equations using the **calculus approach**.

Motion in a Plane

2 Horizontal Projectile Motion

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Horizontal Projectile Motion

In horizontal projectile motion, a body is projected with an initial velocity u in the horizontal direction from a height H. The motion can be analyzed as a combination of two independent components:

1. **Horizontal motion:** Uniform motion with constant velocity u.
2. **Vertical motion:** Uniformly accelerated motion under the influence of gravity g (taken as negative for downward motion).

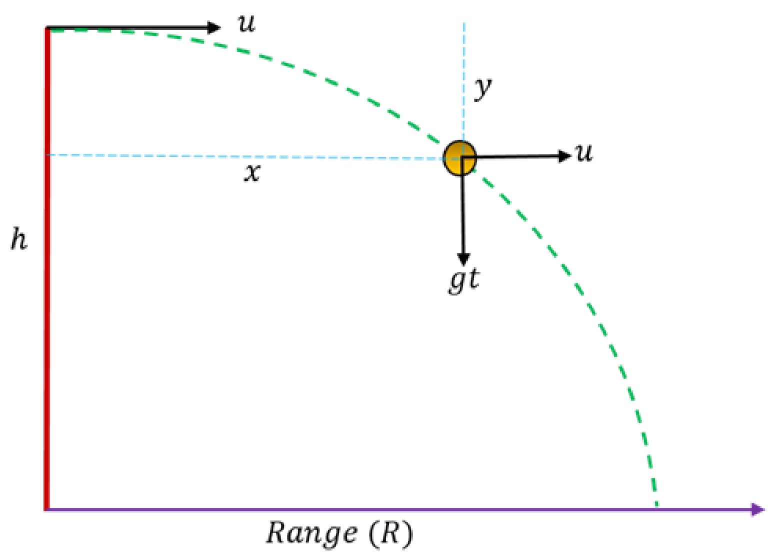

Figure 1: Horizontal projectile motion

Theory:

- The horizontal velocity remains constant because there is no acceleration in the horizontal direction.
- The vertical motion is influenced only by gravity, leading to an acceleration g.
- The trajectory of the body is a parabola, as the horizontal and vertical motions combine to form a curved path.

2.1 Cartesian Equation of Trajectory for Horizontal Projectile Motion

To derive the **trajectory equation** of the horizontal projectile, we eliminate time t from the horizontal and vertical motion equations.

2.1.1 Step 1: Horizontal Displacement Equation

The horizontal displacement is given by:

$$x = ut$$

Solving for t:

$$t = \frac{x}{u}$$

2.1.2 Step 2: Vertical Displacement Equation

The vertical displacement is:

$$y = -\frac{1}{2}gt^2$$

Substituting $t = \frac{x}{u}$ into the vertical motion equation:

$$y = -\frac{1}{2}g\left(\frac{x}{u}\right)^2$$

$$y = -\frac{g}{2u^2}x^2$$

2.1.3 Final Trajectory Equation

The Cartesian equation of the trajectory is:

$$\boxed{y = -\frac{g}{2u^2}x^2}$$

This equation represents a **parabolic trajectory**, confirming that the horizontal projectile follows a parabolic path.

2.2 Time of Flight (T)

The **time of flight** is the total time taken by the body to hit the ground. This depends only on the vertical motion.

From the vertical displacement equation:

$$y = -H$$

Substituting the equation of motion:

$$y = -\frac{1}{2}gT^2$$

Equating:

$$-H = -\frac{1}{2}gT^2$$

Rearranging:

$$T^2 = \frac{2H}{g}$$

Taking the square root on both sides:

$$T = \sqrt{\frac{2H}{g}}$$

$$\boxed{T = \sqrt{\frac{2H}{g}}}$$

Thus, the time of flight depends on the initial height (H) and the acceleration due to gravity (g), but not on the initial velocity (u).

2.3 Horizontal Range (R)

The **horizontal range** is the distance covered in the horizontal direction before the body hits the ground.

Using horizontal motion:

$$x = ut$$

For the total flight time:

$$R = uT$$

Substituting $T = \sqrt{\frac{2H}{g}}$:

$$R = u\sqrt{\frac{2H}{g}}$$

$$\boxed{R = u\sqrt{\frac{2H}{g}}}$$

Thus, the horizontal range depends on the initial velocity (u) and the initial height (H), but not on the gravitational acceleration (g) alone.

2.4 Velocity at Any Instant ($\vec{v}$)

At any time t, the velocity has both horizontal and vertical components.

Horizontal Component:

$$v_x = u$$

Vertical Component:

From vertical motion:

$$v_y = -gt$$

The velocity vector is:

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

Substituting the components:

$$\vec{v} = u\hat{i} + (-gt)\hat{j}$$

Magnitude of Velocity:

The magnitude of velocity is given by:

$$v = \sqrt{v_x^2 + v_y^2}$$

Substituting the components:

$$v = \sqrt{u^2 + (-gt)^2}$$

$$v = \sqrt{u^2 + g^2t^2}$$

Thus, the velocity at any instant depends on both the initial horizontal velocity (u) and the vertical component introduced by gravity (gt).

3 Angular Projectile Motion

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Angular Projectile Motion

A body is projected from the ground with an initial velocity u at an angle θ with the horizontal. The motion can be analyzed using horizontal and vertical components:

- The horizontal velocity component is $u_x = u\cos\theta$.
- The vertical velocity component is $u_y = u\sin\theta$.

- Acceleration due to gravity g acts downward (negative direction).

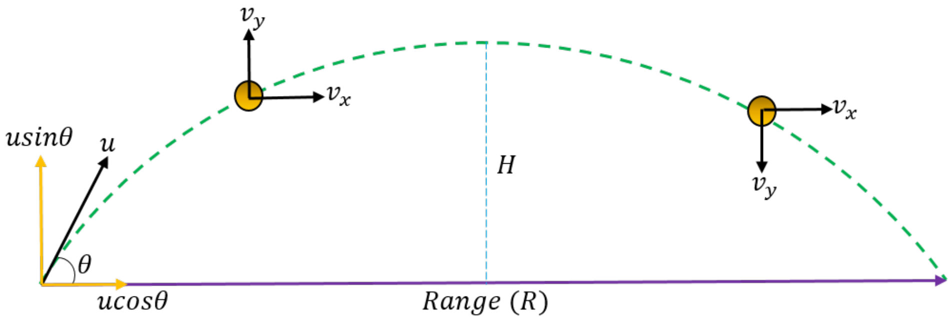

Figure 2: Angular projectile motion

3.1 Cartesian Equation of Trajectory

To derive the trajectory equation, we eliminate t from the horizontal and vertical motion equations.

Step 1: Horizontal Motion Equation

$$x = u_x t = (u\cos\theta)t$$

Solving for t:

$$t = \frac{x}{u\cos\theta}$$

Step 2: Vertical Motion Equation

$$y = u_y t - \frac{1}{2}gt^2$$

Substituting $u_y = u\sin\theta$:

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Step 3: Eliminating t

Substituting $t = \frac{x}{u\cos\theta}$:

$$y = (u\sin\theta)\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

Simplifying:

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2$$

Final Cartesian Equation of Trajectory:

$$\boxed{y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2}$$

This equation represents a **parabolic trajectory**.

3.2 Time of Flight (T)

The time of flight is the total time for which the projectile remains in the air.

Using the vertical motion equation:

$$y = u_y t - \frac{1}{2} g t^2$$

At the highest point and when the projectile lands, $y = 0$:

$$0 = (u \sin\theta) T - \frac{1}{2} g T^2$$

Rearranging:

$$T \left(u \sin\theta - \frac{1}{2} g T \right) = 0$$

Solving for T:

$$T = \frac{2u \sin\theta}{g}$$

Final Time of Flight:

$$\boxed{T = \frac{2u \sin\theta}{g}}$$

3.3 Maximum Height Attained (H)

At maximum height, the vertical velocity becomes zero ($v_y = 0$). Using:

$$v_y^2 = u_y^2 - 2gH$$

Substituting $u_y = u \sin\theta$ and $v_y = 0$:

$$0 = (u \sin\theta)^2 - 2gH$$

Solving for H:

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Final Maximum Height:

$$\boxed{H = \frac{u^2 \sin^2\theta}{2g}}$$

3.4 Horizontal Range (R)

The horizontal range is the total horizontal distance covered during the flight. It is given by:

$$R = u_x T$$

Substituting $u_x = u \cos\theta$ and $T = \frac{2u \sin\theta}{g}$:

$$R = (u \cos\theta) \times \frac{2u \sin\theta}{g}$$

Using the identity $2 \sin\theta \cos\theta = \sin 2\theta$:

$$R = \frac{u^2 \sin 2\theta}{g}$$

Final Horizontal Range:

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

3.5 Velocity at Any Instant ($\vec{v}$)

The velocity at any instant t consists of both horizontal and vertical components.

Horizontal Component:

$$v_x = u \cos\theta$$

Vertical Component:

$$v_y = u \sin\theta - gt$$

Velocity Vector:

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

Substituting values:

$$\vec{v} = u \cos\theta \hat{i} + (u \sin\theta - gt)\hat{j}$$

Magnitude of Velocity:

$$v = \sqrt{v_x^2 + v_y^2}$$

Substituting:

$$v = \sqrt{u^2 \cos^2\theta + (u \sin\theta - gt)^2}$$

Final Magnitude of Velocity:

$$\boxed{v = \sqrt{u^2 + g^2 t^2 - 2ugt \sin\theta}}$$

This formula represents the velocity magnitude at any instant during projectile motion.

Laws of motion

4 Banking of Roads

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Banking of Roads

A vehicle moving on a banked road experiences both **normal reaction** and **friction**, which together provide the necessary centripetal force.

4.1 Forces Acting on the Vehicle

The forces acting on a car of mass m moving on a banked curve of radius r are:

- **Weight** (mg) acting vertically downward.
- **Normal Reaction** (N) exerted by the road, acting perpendicular to the surface.
- **Friction Force** (f) between the tires and the road surface, which acts along the inclined plane.

4.2 Resolving Forces into Components

The forces are resolved into horizontal and vertical components:

- **Vertical direction (force balance equation):**

$$mg + f\sin\theta = N\cos\theta$$

$$mg = N\cos\theta - f\sin\theta$$

- **Horizontal direction (providing centripetal force):**

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

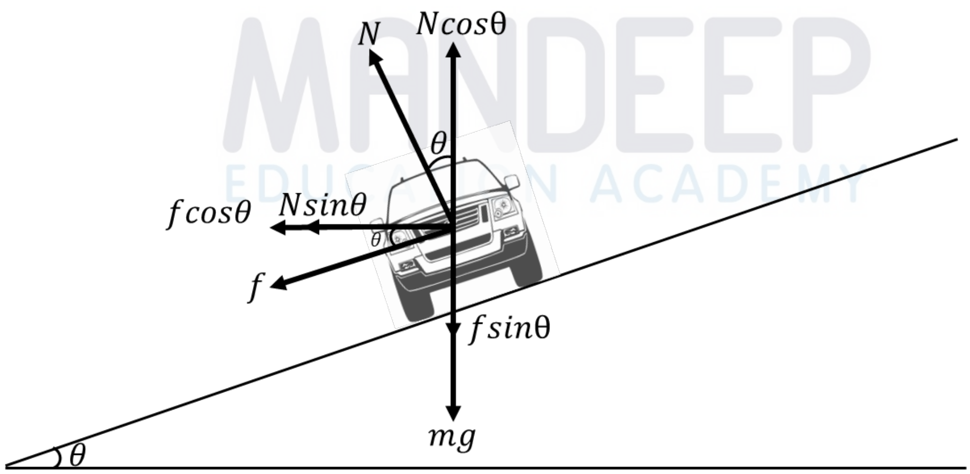

Figure 3: Banking of roads with friction

4.3 Derivation of Maximum Speed for Safe Turn

Dividing the centripetal force equation by the vertical force equation:

$$\frac{N\sin\theta + f\cos\theta}{N\cos\theta - f\sin\theta} = \frac{mv^2}{rmg}$$

$$\frac{v^2}{rg} = \frac{\sin\theta + \frac{f}{N}\cos\theta}{\cos\theta - \frac{f}{N}\sin\theta}$$

Since $\frac{f}{N} = \mu$ (coefficient of friction), we get:

$$\frac{v^2}{rg} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}$$

$$v = \sqrt{\frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}}$$

Final Condition for Safe Turning with Banking and Friction

The maximum speed v at which a vehicle can take a turn safely on a banked road with friction is:

$$v = \sqrt{\frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}}$$

where:

- r = Radius of the turn.
- g = Acceleration due to gravity.
- θ = Banking angle of the road.
- μ = Coefficient of friction between the tires and the road.

Work Energy Power

5 Velocities After One-Dimensional Elastic Collision

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Elastic collision in 1 dimension

Consider two bodies of masses m_1 and m_2 moving along the same straight line with initial velocities u_1 and u_2, respectively. After an elastic collision, their velocities change to v_1 and v_2.

According to the law of conservation of momentum, the total momentum before and after the collision remains the same:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Rearranging,

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad ...(i)$$

Since the collision is elastic, the total kinetic energy is also conserved:

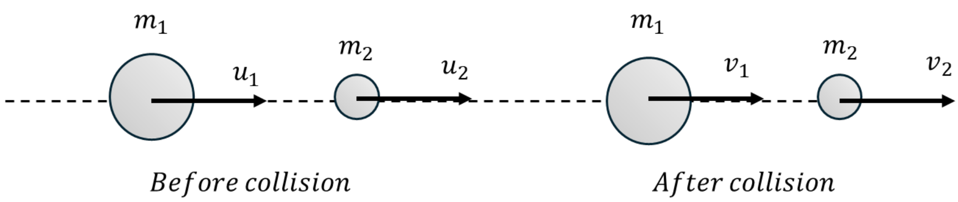

Figure 4: Elastic collision in 1 dimension

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Multiplying the entire equation by 2 to eliminate fractions:

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2$$

Rearranging,

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad \text{...(ii)}$$

We can factorize equation (ii) using the identity $a^2 - b^2 = (a-b)(a+b)$:

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2)$$

Dividing equation (ii) by equation (i), we get:

$$\frac{(u_1 - v_1)(u_1 + v_1)}{(u_1 - v_1)} = \frac{(v_2 - u_2)(v_2 + u_2)}{(v_2 - u_2)}$$

Canceling common terms:

$$u_1 + v_1 = v_2 + u_2$$

Rearranging:

$$u_1 - u_2 = v_2 - v_1 \quad \text{...(iii)}$$

This shows that the **relative velocity of approach** before the collision is equal to the **relative velocity of separation** after the collision.

The coefficient of restitution (e) is given by the formula:

$$e = \frac{\text{relative velocity of separation}}{\text{relative velocity of approach}}$$

For an elastic collision, $e = 1$, so:

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

From the relative velocity condition:

$$v_2 = u_1 - u_2 + v_1$$

Substituting this into the momentum conservation equation:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2(u_1 - u_2 + v_1)$$

Expanding:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2u_1 - m_2u_2 + m_2v_1$$

Rearranging:

$$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$

Solving for v_1:

$$v_1 = \frac{(m_1 - m_2)u_1}{(m_1 + m_2)} + \frac{2m_2u_2}{(m_1 + m_2)}$$

Similarly, solving for v_2:

$$v_2 = \frac{(m_2 - m_1)u_2}{(m_1 + m_2)} + \frac{2m_1u_1}{(m_1 + m_2)}$$

Thus, these equations give the final velocities of the two bodies after a one-dimensional perfectly elastic collision.

System of particles and rotational motion

6 Position Vector of the Center of Mass of a Two-Particle System

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Position vector of centre of mass

Consider a system of two particles P_1 and P_2 with masses m_1 and m_2, respectively. Let their position vectors at any instant t be $\vec{r}_1$ and $\vec{r}_2$, measured with respect to the origin O, as shown in the figure.

The external and internal forces acting on the particles are:

- $\vec{f}_1, \vec{f}_2 \rightarrow$ external forces on particles - $\vec{F}_{12}, \vec{F}_{21} \rightarrow$ internal forces due to mutual interaction

According to Newton's Second Law, the total force acting on the system is:

$$\vec{f}_1 + \vec{f}_2 + \vec{F}_{12} + \vec{F}_{21} = \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2)$$

Since velocity is the time derivative of position,

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt}, \quad \vec{v}_2 = \frac{d\vec{r}_2}{dt}$$

Thus, the equation becomes:

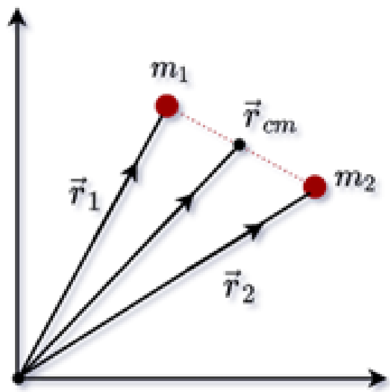

Figure 5: Centre of mass of a two particle system

$$\vec{f_1} + \vec{f_2} + \vec{F}_{12} + \vec{F}_{21} = \frac{d}{dt}\left(m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt}\right)$$

By Newton's Third Law:

$$\vec{F}_{12} = -\vec{F}_{21}$$

So the internal forces cancel out, and we are left with:

$$\vec{f_1} + \vec{f_2} = \frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2)$$

Multiplying and dividing the right-hand side by $m_1 + m_2$:

$$\vec{f_1} + \vec{f_2} = (m_1 + m_2)\frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right)$$

Letting $\vec{F} = \vec{f_1} + \vec{f_2}$, we get:

$$\vec{F} = (m_1 + m_2)\frac{d^2}{dt^2}\vec{R}_{CM}$$

Comparing this equation with:

$$\vec{F} = (m_1 + m_2)\frac{d^2}{dt^2}\vec{R}_{CM}$$

we obtain the position vector of the center of mass:

$$\boxed{\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}}$$

6.1 Generalizing for an n-Particle System

For a system of n particles, the center of mass position vector is given by:

$$\boxed{\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}}$$

This equation shows that the center of mass is the weighted mean of the position vectors of all particles.

Gravitation

7 Variation of Acceleration due to Gravity with Height

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Variation of g with height

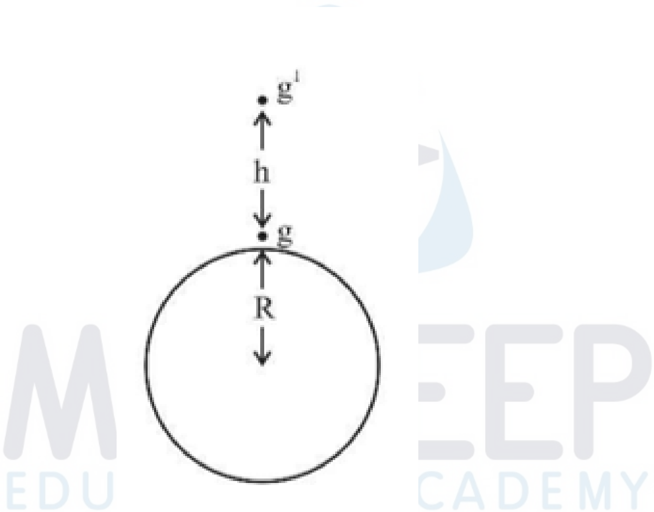

Figure 6: Variation of g with height

The acceleration due to gravity at the surface of the Earth is given by:

$$g = \frac{GM}{R^2}$$

At a height h above the surface of the Earth, the acceleration due to gravity is:

$$g' = \frac{GM}{(R+h)^2}$$

7.1 Derivation of the Expression

Dividing g' by g:

$$\frac{g'}{g} = \frac{\frac{GM}{(R+h)^2}}{\frac{GM}{R^2}}$$

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} \quad \text{(Equation 1)}$$

Rewriting in a simplified form:

$$\frac{g'}{g} = \frac{R^2}{R^2(1+\frac{h}{R})^2}$$

$$\frac{g'}{g} = \frac{1}{(1+\frac{h}{R})^2}$$

7.2 Binomial Approximation for Small h

For $h \ll R$, we use the binomial approximation:

$$(1+x)^n \approx 1 + nx \quad \text{for } x \ll 1$$

Applying $n = -2$ and $x = \frac{h}{R}$:

$$\left(1+\frac{h}{R}\right)^{-2} \approx 1 - \frac{2h}{R}$$

Thus,

$$\frac{g'}{g} \approx 1 - \frac{2h}{R}$$

7.3 Percentage Decrease in Gravity

Rearranging:

$$1 - \frac{g'}{g} = \frac{2h}{R}$$

$$\frac{g-g'}{g} \times 100 = \frac{2h}{R} \times 100$$

7.4 Final Expression for Percentage Decrease

$$\boxed{\text{Percentage decrease in } g = \frac{2h}{R} \times 100}$$

Thus, the acceleration due to gravity decreases as height increases, and for small heights, the decrease is approximately linear with height.

8 Variation of Acceleration due to Gravity with Depth

If a body is taken to a depth d below the surface of the Earth, the acceleration due to gravity at that depth is given by:

$$g' = \frac{GM'}{(R-d)^2}$$

where M' is the mass of the spherical part of the Earth of radius $(R-d)$.

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Variation of g with depth

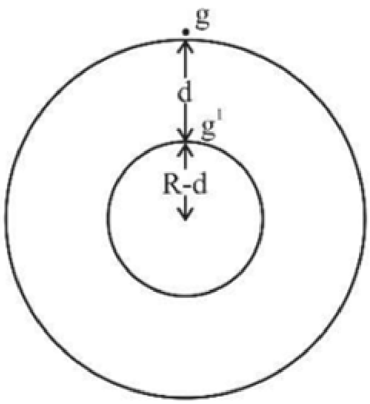

Figure 7: Variation of g with depth

8.1 Derivation of the Expression

Let the Earth be a uniform sphere of density ρ. The total mass of the Earth is:

$$M = \frac{4}{3}\pi R^3 \rho$$

Thus, acceleration due to gravity at the surface of the Earth is:

$$g = \frac{G}{R^2} \times \frac{4}{3}\pi R^3 \rho$$

$$\Rightarrow g = \frac{4}{3} G\pi R\rho$$

Now, the mass of the Earth up to a depth d, i.e., within a sphere of radius $R - d$, is:

$$M' = \frac{4}{3}\pi (R-d)^3 \rho$$

8.2 Acceleration due to Gravity at Depth d

$$g' = \frac{G}{(R-d)^2} \times \frac{4}{3}\pi (R-d)^3 \rho$$

$$\Rightarrow g' = \frac{4}{3} G\pi (R-d)\rho$$

8.3 Ratio of g' to g

Dividing both sides:

$$\frac{g'}{g} = \frac{\frac{4}{3}G\pi(R-d)\rho}{\frac{4}{3}G\pi R\rho}$$

$$\Rightarrow \frac{g'}{g} = \frac{R-d}{R}$$

$$\Rightarrow g' = g\left(1 - \frac{d}{R}\right)$$

8.4 Percentage Decrease in g at Depth d

$$\text{Percentage decrease} = \left(1 - \frac{d}{R}\right) \times 100$$

$$\boxed{\text{Percentage decrease in } g = \frac{d}{R} \times 100}$$

Thus, acceleration due to gravity decreases linearly as depth increases.

Mechanical Properties of Solids

9 Energy Stored in a Deformed Body

When a wire is stretched, interatomic forces come into play which oppose the change in configuration of the wire. Hence, work must be done against these restoring forces. This work done in stretching the wire is stored in it as its **elastic potential energy**.

9.1 Work Done in Stretching a Wire

Let a force F applied to a wire of length L increase its length by ΔL. Initially, the internal restoring force in the wire is zero. As the length increases by ΔL, the internal force increases from 0 to F.

Average internal force during stretching:

$$F_{\text{avg}} = \frac{0+F}{2} = \frac{F}{2}$$

Work done on the wire:

$$W = \text{Average Force} \times \text{Increase in Length}$$

$$W = \frac{F}{2} \times \Delta L$$

This work is stored as **elastic potential energy** U in the wire:

$$U = \frac{1}{2}F \times \Delta L$$

9.2 Elastic Potential Energy in Terms of Stress and Strain

Let A be the area of cross-section of the wire. We use:

$$F = \text{Stress} \times A, \quad \frac{\Delta L}{L} = \text{Strain}$$

$$U = \frac{1}{2}\left(\frac{F}{A}\right) \times \left(\frac{\Delta L}{L}\right) \times (AL)$$

Final Expression for Elastic Potential Energy:

$$\boxed{U = \frac{1}{2} \times \text{Stress} \times \text{Strain} \times \text{Volume of Wire}}$$

9.3 Elastic Energy Density

The **elastic energy density** μ is the elastic potential energy per unit volume:

$$\mu = \frac{U}{\text{Volume}} = \frac{1}{2} \times \text{Stress} \times \text{Strain}$$

Since,

$$\text{Stress} = \text{Young's Modulus} \times \text{Strain}$$

Final Expression for Elastic Energy Density:

$$\boxed{\mu = \frac{1}{2} \times \text{Young's Modulus} \times (\text{Strain})^2}$$

Thus, the energy stored per unit volume in a stretched wire is directly proportional to the square of the strain.

Mechanical Properties of Fluids

10 Bernoulli's Principle

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Bernoulli's theorem

Bernoulli's Principle states that the sum of **pressure energy, kinetic energy, and potential energy per unit volume** of an incompressible, non-viscous fluid in a streamlined irrotational flow remains constant along a streamline. Mathematically, it is given by:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

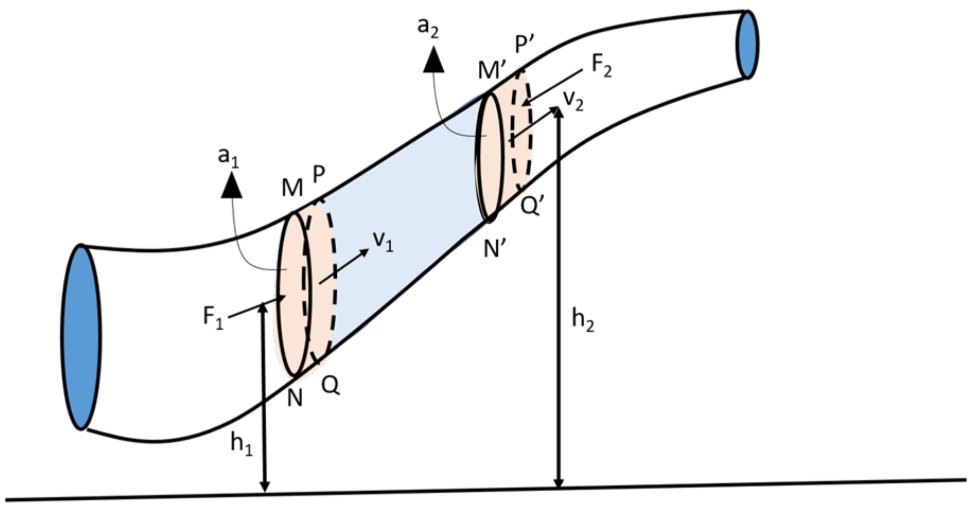

Figure 8: Bernoulli's theorem

10.1 Proof of Bernoulli's Theorem

Consider a non-viscous and incompressible fluid flowing steadily between two sections A and B of a pipe of varying cross-section. Let:

- a_1 and a_2 be the areas of cross-section at A and B, respectively.
- v_1 and v_2 be the fluid velocities at A and B.
- P_1 and P_2 be the fluid pressures at A and B.
- h_1 and h_2 be the heights above the reference level.

Let ρ be the density of the fluid. Since the fluid is incompressible, the mass flow rate remains constant:

$$\text{Mass of fluid in time } \Delta t = \text{Volume} \times \text{Density}$$

$$m = a_1 v_1 \Delta t \rho = a_2 v_2 \Delta t \rho$$

The change in kinetic energy of the fluid is given by:

$$\Delta KE = KE_{\text{at } B} - KE_{\text{at } A}$$

$$= \frac{1}{2} m (v_2^2 - v_1^2)$$

$$= \frac{1}{2} a_1 v_1 \Delta t \rho (v_2^2 - v_1^2)$$

The change in potential energy of the fluid is given by:

$$\Delta PE = PE_{\text{at } B} - PE_{\text{at } A}$$

$$= mg(h_2 - h_1)$$

$$= a_1 v_1 \Delta t \rho g (h_2 - h_1)$$

The work done by the fluid is:

$$\text{Work done at } A - \text{Work done at } B$$

$$= P_1 a_1 v_1 \Delta t - P_2 a_2 v_2 \Delta t$$

$$= a_1 v_1 \Delta t (P_1 - P_2)$$

By the work-energy theorem:

$$\text{Net Work Done} = \text{Change in Kinetic Energy} + \text{Change in Potential Energy}$$

$$a_1 v_1 \Delta t (P_1 - P_2) = \frac{1}{2} a_1 v_1 \Delta t \rho (v_2^2 - v_1^2) + a_1 v_1 \Delta t \rho g (h_2 - h_1)$$

Dividing by $a_1 v_1 \Delta t$, we get:

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 + \rho g h_2 - \rho g h_1$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

$$\boxed{P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$$

Thus, Bernoulli's theorem is proved, showing that the sum of pressure energy, kinetic energy, and potential energy per unit volume remains constant along a streamline.

Thermal Properties of Matter

11 Relation Between Temperature Coefficients of Thermal Expansion

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Relation between alpha, beta and gamma

Thermal expansion occurs in three dimensions and is characterized by three coefficients:

- α – **Coefficient of linear expansion**
- β – **Coefficient of area expansion**
- γ – **Coefficient of volume expansion**

11.1 1. Definition of the Coefficients

(i) Linear Expansion: When a solid is heated, its length increases according to:

$$L = L_0(1 + \alpha\Delta T)$$

where:

- L_0 is the initial length,
- L is the final length after temperature rise ΔT,
- α is the coefficient of linear expansion.

(ii) Area Expansion: The change in area of a solid is given by:

$$A = A_0(1 + \beta\Delta T)$$

where A_0 is the initial area and β is the coefficient of area expansion.
(iii) Volume Expansion: The change in volume of a solid follows:

$$V = V_0(1 + \gamma\Delta T)$$

where V_0 is the initial volume and γ is the coefficient of volume expansion.

11.2 2. Derivation of the Relation

Since area is the product of two lengths, we approximate:

$$A = L_1 \times L_2$$

Using the linear expansion formula:

$$L_1 = L_{10}(1 + \alpha\Delta T), \quad L_2 = L_{20}(1 + \alpha\Delta T)$$

$$A = L_{10}(1 + \alpha\Delta T) \times L_{20}(1 + \alpha\Delta T)$$

Expanding using binomial approximation for small $\alpha\Delta T$:

$$A = A_0(1 + 2\alpha\Delta T + \alpha^2\Delta T^2)$$

Since $\alpha^2\Delta T^2$ is very small, it is neglected:

$$A = A_0(1 + 2\alpha\Delta T)$$

Comparing with the area expansion formula:

$$1 + \beta\Delta T = 1 + 2\alpha\Delta T$$

$$\Rightarrow \beta = 2\alpha$$

11.3 3. Relation Between α, β, γ

Since volume is the product of three lengths:

$$V = L_1 L_2 L_3$$

Using the linear expansion formula:

$$V = L_{10}(1 + \alpha\Delta T) \times L_{20}(1 + \alpha\Delta T) \times L_{30}(1 + \alpha\Delta T)$$

Expanding:

$$V = V_0(1 + 3\alpha\Delta T + 3\alpha^2\Delta T^2 + \alpha^3\Delta T^3)$$

Neglecting higher-order small terms:

$$V = V_0(1 + 3\alpha\Delta T)$$

Comparing with the volume expansion formula:

$$1 + \gamma\Delta T = 1 + 3\alpha\Delta T$$

$$\Rightarrow \gamma = 3\alpha$$

11.4 4. Final Relationship

Thus, the required relations between the temperature coefficients of thermal expansion are:

$$\boxed{\beta = 2\alpha, \quad \gamma = 3\alpha}$$

These equations show that the area expansion coefficient is twice the linear expansion coefficient, and the volume expansion coefficient is three times the linear expansion coefficient.

Thermodynamics

12 Molar Heat Relation: Mayer's Formula

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Mayer's formula

Consider n moles of an ideal gas. Heat is supplied to raise its temperature by dT. According to the **first law of thermodynamics**, the heat supplied dQ is used partly to increase the **internal energy** and partly to do the **work of expansion**. That is:

$$dQ = dU + PdV$$

If the heat dQ is absorbed at **constant volume**, then $dV = 0$, so:

$$dQ = nC_v dT$$

$$\Rightarrow dQ = dU$$

$$\Rightarrow dU = nC_v dT \quad \cdots \textbf{(i)}$$

If the heat dQ is absorbed at **constant pressure**, then:

$$dQ = dU + PdV$$

$$\Rightarrow nC_p dT = dU + PdV$$

Since the **change in internal energy is the same in both cases** (because the temperature change is the same), we substitute equation **(i)**:

$$nC_p dT = nC_v dT + PdV$$

$$\Rightarrow n(C_p - C_v)dT = PdV$$

For an ideal gas,

$$PV = nRT$$

Differentiating both sides:

$$PdV = nRdT$$

Substituting $PdV = nRdT$ in the previous equation:

$$n(C_p - C_v)dT = nRdT$$

Dividing by ndT:

$$C_p - C_v = R$$

$$\boxed{C_p - C_v = R}$$

Thus, this is the required **Mayer's Formula**, which establishes the relation between molar heat capacities at constant pressure and constant volume.

Kinetic Theory of Gases

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Pressure due to an ideal gas

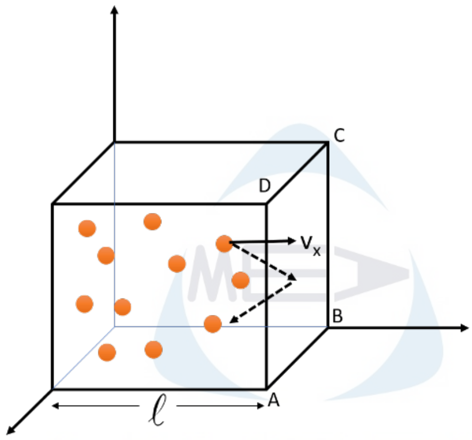

Figure 9: Pressure due to ideal gas

13 Pressure Due to an Ideal Gas

Consider a cubical chamber of edge length ℓ containing an ideal gas. Let the number of molecules per unit volume be n. Consider a molecule with velocity v having components v_x, v_y, v_z.

The momentum of a gas molecule before hitting a wall (say ABCD) is:

$$\text{Initial Momentum} = mv_x$$

Since collisions of an ideal gas (according to KTG) are perfectly elastic, the molecule rebounds with velocity $-v_x$ after hitting the wall. Hence, the momentum after collision is:

$$\text{Final Momentum} = -mv_x$$

Change in momentum of the molecule:

$$\Delta p = -mv_x - mv_x = -2mv_x$$

Momentum imparted to the wall:

$$\text{Momentum imparted} = 2mv_x$$

The number of molecules that can hit the wall in time Δt is given by:

$$n\bar{v}_x \Delta t \ell^2$$

However, since half of the molecules are moving in the opposite direction, the number of molecules that actually hit the wall is:

$$\frac{1}{2}n\bar{v}_x\Delta t\ell^2$$

Total momentum transferred to the wall in time Δt:

$$\text{Total Momentum} = \frac{1}{2}n\bar{v}_x\Delta t\ell^2 \times 2m\bar{v}_x$$

$$= mn\bar{v}_x^2\Delta t\ell^2$$

Force exerted on the wall:

$$F = \frac{\text{Total Momentum}}{\Delta t}$$

$$F = \frac{mn\bar{v}_x^2\Delta t\ell^2}{\Delta t}$$

$$F = mn\bar{v}_x^2\ell^2$$

Since pressure is force per unit area:

$$P_x = \frac{\text{Force}}{\text{Area}}$$

$$P_x = \frac{mn\bar{v}_x^2\ell^2}{\ell^2} = mn\bar{v}_x^2$$

Since gas molecules move randomly in all directions, the velocity components are equal in all directions:

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2$$

$$\bar{v}^2 = 3\bar{v}_x^2 \quad \Rightarrow \quad \bar{v}_x^2 = \frac{1}{3}\bar{v}^2$$

Thus, substituting in the pressure equation:

$$P = \frac{1}{3}mn\bar{v}^2$$

Since $mn = \rho$ (density of gas), we get:

$$\boxed{P = \frac{1}{3}\rho\bar{v}^2}$$

This equation represents the pressure exerted by an ideal gas in terms of molecular velocity.

Oscillations

14 Simple Harmonic Motion (SHM)

A body executing SHM can be compared to a body performing uniform circular motion. Let the body move in a circular path of radius A and cover an angle θ in time t after starting from $X(A, 0)$ at $t = 0$.

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Displacement equation in SHM

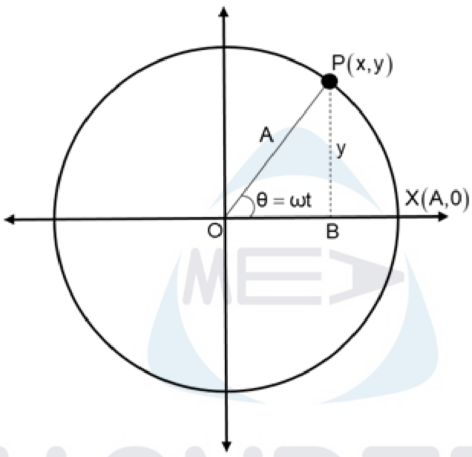

Figure 10: Comparison of circular motion with SHM

14.1 Displacement in SHM

From the right-angled triangle,

$$\frac{y}{A} = \sin\theta$$

Since $\theta = \omega t$,

$$y = A\sin\omega t$$

This is the displacement equation of a body whose motion has amplitude A and angular frequency ω.

14.2 Velocity in SHM

Velocity is the time derivative of displacement:

$$v = \frac{dy}{dt} = A\omega\cos\omega t$$

Using $\sin^2\theta + \cos^2\theta = 1$:

$$v = A\omega\sqrt{1 - \sin^2\omega t}$$

$$v = A\omega\sqrt{1 - \frac{y^2}{A^2}}$$

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Velocity & acceleration in SHM

$$v = \omega\sqrt{A^2 - y^2}$$

The maximum velocity occurs at $y = 0$:

$$v_{\text{max}} = A\omega$$

14.3 Acceleration in SHM

Acceleration is the derivative of velocity:

$$a = \frac{dv}{dt} = \frac{d}{dt}(A\omega\cos\omega t)$$

$$a = -A\omega^2\sin\omega t$$

$$a = -\omega^2 y$$

14.4 Restoring Force in SHM

By Newton's Second Law,

$$F = ma$$

$$F = -m\omega^2 y$$

14.5 Time Period of SHM

We know that the angular frequency is given by:

$$\omega = \sqrt{\frac{k}{m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Also, from $a = -\omega^2 y$,

$$\omega = \sqrt{\frac{a}{y}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{a}{y}}$$

$$T = 2\pi\sqrt{\frac{y}{a}}$$

14.6 Kinetic Energy in SHM

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Energy in SHM

Kinetic energy is given by:

$$KE = \frac{1}{2}mv^2$$

$$KE = \frac{1}{2}m(\omega\sqrt{A^2 - y^2})^2$$

$$KE = \frac{1}{2}m\omega^2 A^2 - \frac{1}{2}m\omega^2 y^2$$

14.7 Potential Energy in SHM

Potential energy is given by:

$$PE = \frac{1}{2}ky^2$$

Using $k = m\omega^2$,

$$PE = \frac{1}{2}m\omega^2 y^2$$

14.8 Total Energy in SHM

$$TE = PE + KE$$

$$TE = \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}m\omega^2 A^2 - \frac{1}{2}m\omega^2 y^2$$

$$TE = \frac{1}{2}m\omega^2 A^2$$

Thus, total energy in SHM remains constant and is proportional to the square of the amplitude.

$$\boxed{TE = \frac{1}{2}m\omega^2 A^2}$$

Waves

15 Equation of a Plane Progressive Wave

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Equation of plane progressive wave

Suppose a simple harmonic wave starts from the origin O and travels along the positive direction of the X-axis with speed v. Let time t be measured from the instant when the particle at the origin O is passing through the mean position.

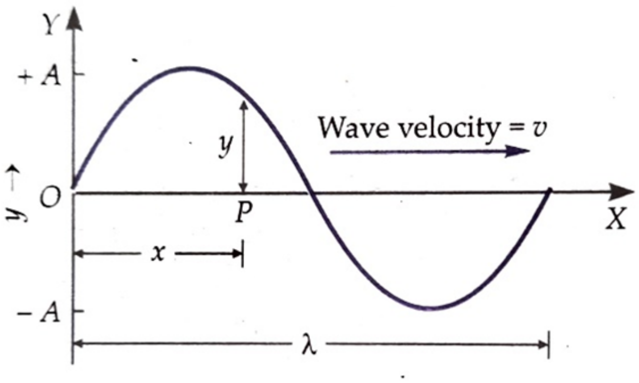

Figure 11: Plane progressive wave

Taking the initial phase of the particle to be zero, the displacement of the particle at the origin O (i.e., at $x = 0$) at any instant t is given by:

$$y(0, t) = A \sin \omega t \quad \cdots \textbf{(i)}$$

where:

- A is the **amplitude** of the wave,
- T is the **time period** of the wave.

15.1 Displacement at a Distance x

Consider a particle P on the x-axis at a distance x from O. The disturbance starting from the origin O will reach P in $\frac{x}{v}$ seconds later than the particle at O.

Thus, the displacement of the particle at P at any instant t is the same as the displacement of the particle at O at an earlier time:

$$\text{Displacement at } P = \text{Displacement at } O \text{ at time } \left(t - \frac{x}{v}\right)$$

Using equation (i):

$$y(x,t) = A \sin \omega \left(t - \frac{x}{v}\right)$$

Expanding:

$$y(x,t) = A \sin \left(\omega t - \frac{\omega}{v} x\right)$$

15.2 Wave Number and Final Wave Equation

The term $\frac{\omega}{v}$ can be rewritten using:

$$\frac{\omega}{v} = \frac{2\pi v}{v\lambda} = \frac{2\pi}{\lambda} = k$$

where k is the **angular wave number**, defined as:

$$k = \frac{2\pi}{\lambda}$$

Thus, the final equation of a plane progressive wave is:

$$\boxed{y(x,t) = A \sin(\omega t - kx)}$$

where:

- k is the **wave number**,
- λ is the **wavelength**.

This equation represents a sinusoidal wave moving in the positive x-direction.

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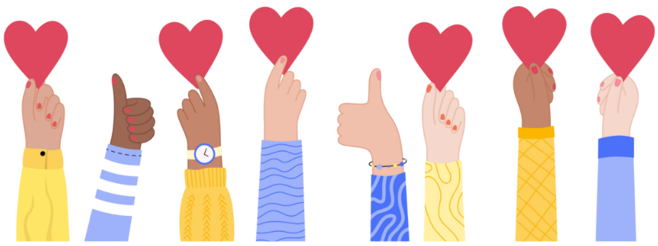

Mandeep Sharma

UPI ID: 8130030691@paytm

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