Motion in a plane

Important question – answers

Question (1): What is a vector quantity? Define the following: Unit vector: Collinear vector or parallel vectors, Polar and axial vector, Coplanar vectors.

A vector quantity is a physical quantity that has both magnitude (size) and direction. It is represented graphically as an arrow, where the length of the arrow represents the magnitude, and the direction of the arrow represents the direction of the quantity being described.

Unit vector: A unit vector is a vector that has a magnitude of 1 and is used to specify a direction in a particular coordinate system. Unit vectors are often denoted by a lowercase letter with a circumflex (hat) symbol, such as $\hat{i}, \hat{j}, \hat{k}$, in a Cartesian coordinate system.

Collinear vector or parallel vectors: Collinear vectors, also known as parallel vectors, are vectors that lie along the same straight line or are multiples of each other. They may have the same or opposite directions but share the same line of action. If two vectors are collinear, it means they are parallel and may have the same or different magnitudes.

Polar and axial vector:

- **Polar vector**: A polar vector is a vector that follows the rules of ordinary vector algebra. It has a definite direction in space and is independent of the coordinate system used to describe it. Examples of polar vectors include displacement, velocity, and force.
- **Axial vector**: An axial vector, also known as a pseudo vector, is a vector that does not follow the ordinary rules of vector algebra under coordinate transformations. Its direction changes under reflection but its magnitude remains the same. Examples of axial vectors include angular velocity, torque, and magnetic field.

Coplanar vectors: Coplanar vectors are vectors that lie in the same plane. In three-dimensional space, any two vectors are coplanar if they can be represented within the same plane. If three or more vectors are coplanar, they all lie in the same plane.

Questions (2): Define triangle law of vector addition by giving suitable examples.

The Triangle Law of Vector Addition is a rule used to find the resultant of two vectors. According to this law, if two vectors are represented as two sides of a triangle taken in order, then the resultant vector is represented by the closing side of the triangle taken in the opposite order.

Definition: If two vectors $\vec{A}$ and $\vec{B}$ are represented by two sides OA and AB of a triangle OAB taken in order, then the resultant vector $\vec{R} = \vec{A} + \vec{B}$ is represented in magnitude and direction by the side OB of the triangle taken in the reverse order.

Example:

Let's consider two vectors, $\vec{A}$ and $\vec{B}$, where $\vec{A}$ represents a displacement of 3 units due east, and $\vec{B}$ represents a displacement of 4 units due north.

1. To use the triangle law, we first draw vector $\vec{A}$ on a graph paper starting from a point O towards the east.
2. From the head of vector $\vec{A}$, we draw vector $\vec{B}$ towards the north.
3. The starting point O of vector $\vec{A}$ and the finishing point of vector $\vec{B}$ are connected to form the third side of the triangle, OB, which is the resultant vector $\vec{R}$.

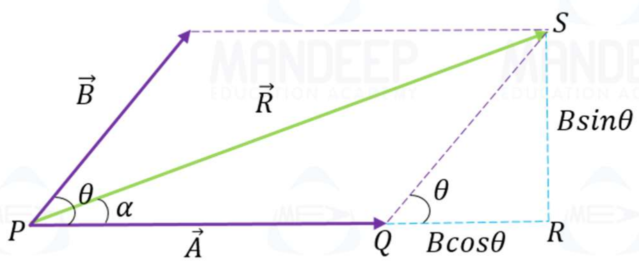
In this example, the triangle formed is a right-angled triangle with sides of 3 units and 4 units. According to the Pythagorean theorem, the length of the resultant vector $\vec{R}$ (the hypotenuse in this case) can be calculated as: $|\vec{R}| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$ units

The direction of $\vec{R}$ with respect to the east (the direction of $\vec{A}$) can be determined using trigonometry, specifically the tangent function, if needed.

Question (3): Discuss parallelogram law of vector addition and derive a formula to find magnitude and resultant of two vectors A and B inclined at an angle θ .

According to the parallelogram law of vector addition: If two vectors are considered to be the adjacent sides of a parallelogram, then the resultant of the two vectors is given by the vector that is diagonal passing through the point of contact of the two vectors.

Consider two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ as shown. Let their resultant be $\vec{R}$.



Extend PQ and draw QR such that $SR \perp QR$.

In ΔQRS

$$\frac{QR}{QS} = \cos\theta \Rightarrow QR = QS\cos\theta = B\cos\theta \qquad \text{.........(i)}$$

$$\frac{SR}{QS} = \sin\theta \Rightarrow SR = QS\sin\theta = B\sin\theta \qquad \text{..........(ii)}$$

In ΔPSR

$$(PS)^2 = (PR)^2 + (SR)^2$$

$$\Rightarrow R^2 = (PQ + QR)^2 + (SR)^2$$

$$\Rightarrow R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2 \qquad \text{[using (i) and (ii)]}$$

$$\Rightarrow R^2 = A^2 + B^2\sin^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow R^2 = A^2 + B^2\left(\sin^2\theta + \cos^2\theta\right) + 2AB\cos\theta$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow \boxed{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then

$$\tan\alpha = \frac{SR}{PR} = \frac{SR}{PQ + QR}$$

$$\Rightarrow \tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

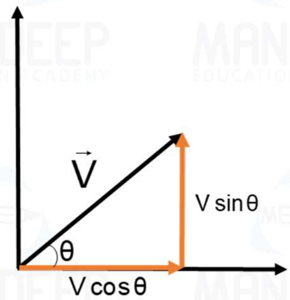
$$\Rightarrow \boxed{\alpha = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)}$$

Question (4): Explain resolution of vector into its rectangular components.

A vector V can be resolved into two components. These components are also vectors and they are perpendicular to each other and they show the effect of the original vectors in each direction.

Give figure shows a vector V. Its two components are these:

(i) Vcosθ in the horizontal direction and
(ii) Vcos(90 - θ) = V sinθ in the vertical direction.



Vector v can be written as

$$\vec{V} = V\cos\theta\hat{i} + V\sin\theta\hat{j}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j}$$

Question (5): Explain dot product and cross product of vectors.

Product of two vectors is called dot product when their product gives a scalar quantity. For example, product of force (vector) and displacement (vector) gives work (scalar). So, work is the dot (or scalar) product of force and displacement.

Consider two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ. Then their dot product is.

$$\boxed{\vec{A}.\vec{B} = AB\cos\theta}$$

Clearly,

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$$

Let $\vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, therefore

$$\vec{A}.\vec{B} = a_1a_2\left(\hat{i}.\hat{i}\right) + a_1b_2\left(\hat{i}.\hat{j}\right) + a_1c_2\left(\hat{i}.\hat{k}\right) + b_1a_2\left(\hat{j}.\hat{i}\right) + b_2b_1\left(\hat{j}.\hat{j}\right) + b_1c_2\left(\hat{j}.\hat{k}\right) + c_1a_2(\hat{k}.\hat{i}) + c_1b_2\left(\hat{k}.\hat{j}\right) + c_1c_2\left(\hat{k}.\hat{k}\right)$$

$$\boxed{\vec{A}.\vec{B} = a_1a_2 + b_1b_2 + c_1c_2}$$

Question (6): Explain cross product of vectors.

Definition: The cross product of two vectors $\vec{A}$ and $\vec{B}$ in three-dimensional space is denoted by $\vec{A}\times\vec{B}$. It results in a new vector that is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Formula: The cross product of two vectors $\vec{A}$ and $\vec{B}$ is given by: $\vec{A}\times\vec{B} = \left(AB\sin\theta\right)\hat{n}$

- A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively.
- *θ* is the angle between $\vec{A}$ and $\vec{B}$.
- $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, determined by the right-hand rule.

Direction of cross product is obtained by right hand rule which states that

Pointing the index finger of your right hand in the direction of the first vector, and the middle finger in the direction of the second vector, your thumb will point in the direction of the resulting cross product vector.

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$$
$$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\hat{k}\times\hat{i}=\hat{j}$$
$$\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j}$$

Let $\vec{A}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{B}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$, therefore

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=(b_1c_2-c_1b_2)\hat{i}-(a_1c_2-c_1a_2)\hat{j}+(a_1b_2-b_1a_2)\hat{k}$$

Question (7): Explain two examples from daily life using components of vectors.

(1) Driving a Car:

Acceleration: Components of the acceleration vector affect how fast the car speeds up and changes direction when navigating traffic.

Turning: Vector components of velocity determine how the car's direction changes during turns.

(2) Playing Sports:

Basketball Shot: Velocity components influence the trajectory and accuracy of the ball when shooting.

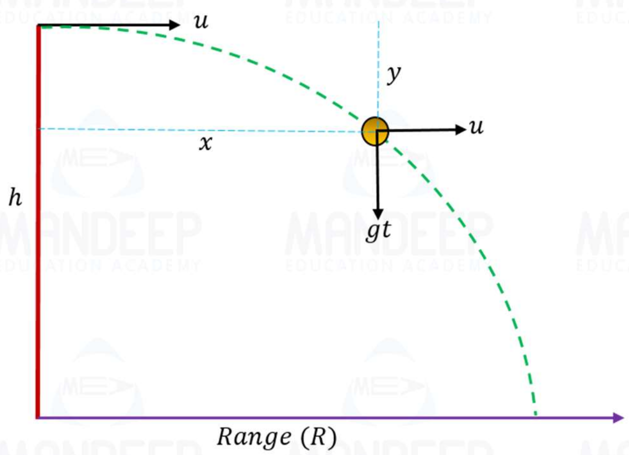
Soccer Kick: Force components affect the speed and direction of the ball when kicking, enabling players to control their passes and shots.

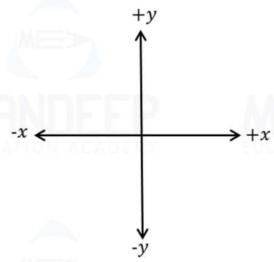
Question (8): What is a projectile. Derive a relation for following quantities when a projectile is fired with velocity u from a height h in the horizontal direction: a. Equation of path b. Time of flight c. Velocity at any instant d. Range

Consider a projectile thrown with velocity u in horizontal direction from a height h as shown

Therefore,

$$u_x=u,\quad u_y=0$$
$$a_x=0,\quad a_y=-g$$





Equation of path

$$x = u_x t + \frac{1}{2}a_x t^2 \Rightarrow \boxed{x = ut}$$

$$y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow \boxed{y = -\frac{1}{2}gt^2}$$

$$\because t = \frac{x}{u}$$

$$y = -\frac{1}{2}g\left(\frac{x}{u}\right)^2 = -\frac{1}{2}\frac{g}{u}x^2$$

Which is a quadratic equation. Thus, path of a projectile is parabolic in nature.

Time of flight

Total time for which the projectile remains in air is called time of flight.

$$\because y = u_y t + \frac{1}{2}a_y t^2$$

$$\therefore -h = (0)t - \frac{1}{2}gT^2$$

$$\Rightarrow \boxed{T = \sqrt{\frac{2h}{g}}}$$

Horizontal range (R)

Maximum horizontal distance travelled by projectile.

$$\because x = u_x t + \frac{1}{2}a_x t^2$$

$$\therefore R = uT + \frac{1}{2}(0)T^2$$

$$\Rightarrow \boxed{R = u\sqrt{\frac{2h}{g}}}$$

$$\therefore -h = (0)t - \frac{1}{2}gT$$

Velocity at any instant

$$v_x = u_x + a_x t$$

$$\Rightarrow \boxed{v_x = u}$$

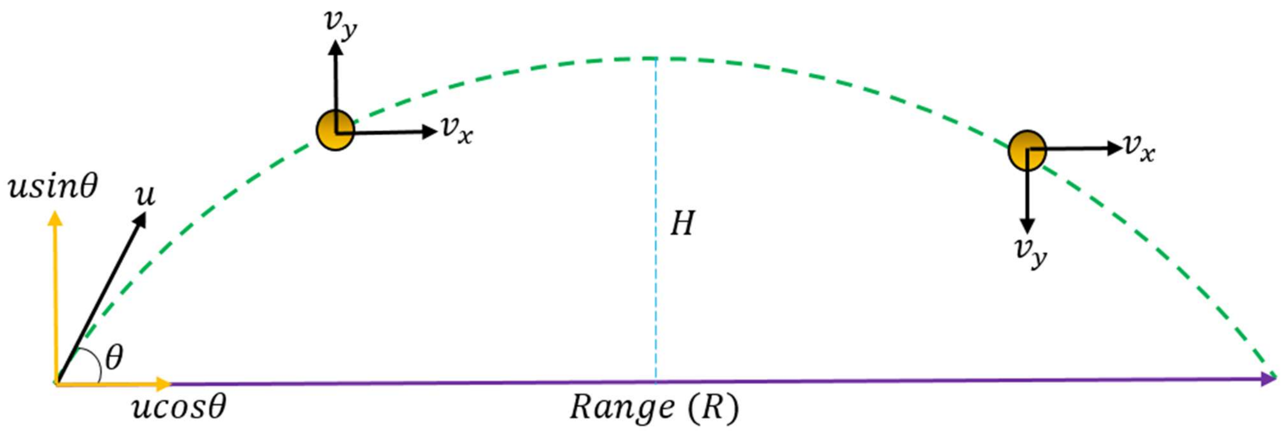
$$v_y = u_y + a_y t$$

$$\Rightarrow \boxed{v_y = -gt}$$

$$\text{As } v = \sqrt{v_x^2 + v_y^2}$$

$$\boxed{v = \sqrt{u^2 + g^2t^2}}$$

Question (9): Derive a relation for following quantities when a projectile is fired with velocity u with horizontal at an angle θ . a. Equation of path b. Time of flight c. Maximum height attained d. Velocity at any instant e. Range



Angular projectile motion

Consider a body projected with velocity u at an angle θ with horizontal as shown.

Therefore,

$$u_x = u\cos\theta,\ \ u_y = u\sin\theta$$
$$a_x = 0 \quad a_y = -g$$

Equation of path

$$x = u_x t + \frac{1}{2}a_x t^2 \Rightarrow \boxed{x = u\cos\theta t}$$

$$y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow \boxed{y = u\sin\theta t - \frac{1}{2}gt^2}$$

$$\because t = \frac{x}{u\cos\theta}$$

$$\therefore y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\Rightarrow \boxed{y = x\tan\theta - \frac{1}{2}\left(\frac{g}{u^2\cos^2\theta}\right)x^2}$$

Time of flight

Total time for which the projectile remains in air is called time of flight.

$$\because y = u_y t + \frac{1}{2}a_y t^2$$

$$\therefore 0 = (u\sin\theta)T - \frac{1}{2}gT^2 \quad [y = 0 \text{ when body hits the ground}]$$

$$\Rightarrow \boxed{T = \frac{2u\sin\theta}{g}}$$

Maximum height attained

At maximum height

$$v_y = 0$$

$$\therefore 0 = u_y + a_y t$$

$$\Rightarrow 0 = u\sin\theta - gt$$

$$\Rightarrow t = \frac{u\sin\theta}{t}$$

Putting this value in equation of y, we get

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\Rightarrow H = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$\Rightarrow H = \frac{u^2\sin^2\theta}{g} - \frac{1}{2}\frac{u^2\sin^2\theta}{g} \Rightarrow \boxed{H = \frac{u^2\sin^2\theta}{2g}}$$

Horizontal range

$$\because x = u_x t + \frac{1}{2}a_x t^2$$

$$R = u\cos\theta\left(\frac{2u\sin\theta}{g}\right) + \frac{1}{2}(0)t^2$$

$$\Rightarrow R = \frac{2u^2\sin\theta\cos\theta}{g}$$

$$\Rightarrow \boxed{R = \frac{u^2\sin 2\theta}{g}}$$

Velocity at any instant

$$v_x = u_x + a_x t = u\cos\theta$$

$$v_y = u_y + a_y t = u\sin\theta - gt$$

$$\because v = \sqrt{v_x^2 + v_y^2}$$

$$\Rightarrow v = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$$

$$\Rightarrow v = \sqrt{u^2\sin^2\theta + u^2\cos^2\theta + g^2t^2 - 2u\sin\theta gt}$$

$$\Rightarrow v = \sqrt{u^2\left(\sin^2\theta + \cos^2\theta\right) + g^2t^2 - 2u\sin\theta gt}$$

$$\Rightarrow \boxed{v = \sqrt{u^2 + g^2t^2 - 2u\sin\theta gt}}$$

Question (10): Show that range for two complimentary angles is same when the projectile is fired with same velocity at these two angles.

Since $R=\dfrac{u^2\sin 2\theta}{g}$

Putting $(90-\theta)$ in place of θ, we get

$$R'=u^2\frac{\sin 2(90-\theta)}{g}=\frac{u^2\sin(180-2\theta)}{g}=\frac{u^2\sin 2\theta}{g}=R$$

Question (11): What is uniform and non-uniform circular motion.

Uniform circular motion and non-uniform circular motion describe the movement of an object along a circular path, but they differ in terms of the object's speed.

Uniform Circular Motion:

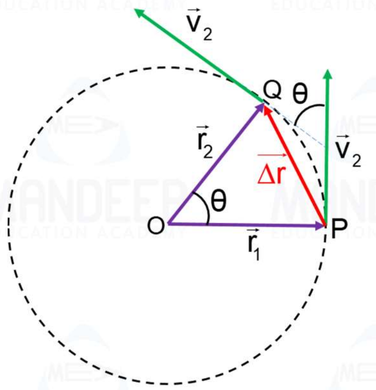
- In uniform circular motion, the object travels around the circular path at a constant speed.
- Despite moving at a constant speed, the object's velocity is not constant because its direction continuously changes, always tangent to the circle.
- Examples include the motion of planets around the sun in nearly circular orbits or the hands of a clock moving at a constant angular speed.

Non-uniform Circular Motion:

- In non-uniform circular motion, the object travels around the circular path at a changing speed.
- This means that the object's velocity magnitude or direction, or both, vary as it moves along the circular path.
- Examples include a car driving around a curved road at varying speeds or a satellite orbiting a planet with varying orbital speeds due to gravitational effects.

Question (12): What is centripetal acceleration. Derive an expression for it.

Consider a body moving in a circle of radius r with velocity v. Let the position vector of body be $\vec{r}_1$ when it is at P and $\vec{r}_2$ when it is Q. The velocity vector of body at P is $\vec{v}_1$ and Q it is $\vec{v}_2$. If angle between $\vec{r}_1$ and $\vec{r}_2$ is θ then clearly angle between $\vec{v}_1$ and $\vec{v}_2$ is also θ.



Clearly $|\vec{r}_1|=|\vec{r}_2|=r$

Since the motion is uniform so, $|\vec{v}_1|=|\vec{v}_2|=v$

Now in

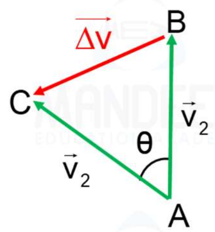
ΔQOP and ΔCAB

$\dfrac{OP}{AB}=\dfrac{OQ}{AC}=\dfrac{r}{v}$

and $\angle QOP=\angle CAB$

$\therefore$ by SAS similarity rule

$\Delta QOP\sim\Delta CAB$



so, $\dfrac{\Delta v}{\Delta r}=\dfrac{v}{r}\Rightarrow\dfrac{\Delta v}{\Delta t}=\dfrac{v}{r}\left(\dfrac{\Delta r}{\Delta t}\right)$

$\Rightarrow \boxed{a_c=\dfrac{v^2}{r}}$